1 Basic Math

1.1 Norm

1. ℓ_p-norm : $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$, $\|x\|_\infty = \max_{1\le i\le n} |x_1|$.
2. **dual norm** for $\|\cdot\|$ is $\|x\|_* = \max_{\|y\|\le 1} y^\top x$.
3. The dual norm for ℓ_p is ℓ_q where $1/p + 1/q = 1$.

1.2 Inequalities

1. **Cauchy–Schwarz Inequality:** $|\langle \mathbf{u}, \mathbf{v}\rangle| \le \|\mathbf{u}\| \cdot \|\mathbf{v}\|$ $\left(\int_a^b |f(x)g(x)|\,dx\right)^2 \le \left(\int_a^b |f(x)|^2\,dx\right)\left(\int_a^b |g(x)|^2\,dx\right)$
2. **Holder Inequality:** $\int_a^b |f(x)g(x)|\,dx \le \|f\|_p \cdot \|g\|_q$ for $1/p + 1/q = 1$.
3. **Markov's Inequality:** $\mathbb{P}(X \ge a) \le \frac{\mathbb{E}[X]}{a}$ for $a > 0$.
4. **Chernoff Inequality:** $\mathbb{P}(X \ge a) \le \inf_{t>0} M(t)e^{-ta}$ for $a > 0$.

1.3 Matrix

1. **Orthogonal matrices:** A square matrix Q is orthogonal if $Q^\top Q = QQ^\top = I$.
   - $\det Q = \pm 1$
   - Its eigenvalues are placed on the unit circle. (so if real, $\lambda = \pm 1$)
   - Its eigenvectors are unitary, i.e. have length one.
   - Its columns as well as rows form an unit orthonormal set. (Thus, the elements of Q are no larger than 1 in absolute value.)
   - Norm and inner-product preservation
2. **Symmetric matrices:** A square matrix A is symmetric if $A = A^\top$.
   - The **eigenvalues** of a real symmetric matrix are all **real**, and their associated eigenvectors are **orthogonal** to each other.
   - $\sigma_i(A) = \sqrt{\lambda_i(A)} = |\lambda_i(A)|$. $\|A\|_2 = \max_i |\lambda_i(A)|$
3. **Schur complement:** Let $A = \begin{bmatrix} X & Y \\ Y^\top & Z \end{bmatrix}$ be an $n \times n$ real symmetric matrix.
   - Suppose Z is invertible and define $S_A = X - YZ^{-1}Y^\top$. If $Z \in\succ 0$, then $A \succeq 0$ iff $X \succeq 0$ and $S_A \succeq 0$.
   - Suppose X is invertible and define $S_A' = Z - Y^\top X^{-1} Y$. If $X \in\succ 0$, then $A \succeq 0$ iff $Z \succeq 0$ and $S_A' \succeq 0$.
4. **Positive Semidefinite Matrix:** $A \succeq 0$ iff $\forall x \in \mathbb{R}^n$, $x^\top Ax \ge 0$ iff all eigenvalues are non-negative iff there exists a unique $n \times n$ positive semidefinite matrix $A^{1/2}$ such that $A = A^{1/2}A^{1/2}$ iff $\exists B \in \mathbb{R}^{k\times n}$ $(k = rank(A))$ such that $A = B^\top B$
5. **Courant–Fischer theorem** The k-th largest eigenvalue of a symmetric matrix A is given by

$$\lambda_k = \min_{w^1,\ldots,w^{k-1}\in\mathbb{R}^n} \max_{x\perp w^1,\ldots,w^{k-1}} \frac{x^T Ax}{x^T x}$$
$$= \max_{w^1,\ldots,w^{n-k}\in\mathbb{R}^n} \min_{x\perp w^1,\ldots,w^{n-k}} \frac{x^T Ax}{x^T x}.$$

1.3.1 Matrix Decomposition

1. **SVD:** For $A \in \mathbb{R}^{m\times n}$, there $A = U\Lambda V^\top = \sum_{i=1}^r \sigma^i u^i (v^i)^\top$, where $\Lambda_1 \ge \Lambda_2 \ge \ldots > \Lambda_{r+1} = \ldots = \Lambda_q = 0$, $q = \min\{m, n\}$.
   - $\sigma^i(A) = \sqrt{\lambda^i(A^\top A)}$
   - $U \in \mathbb{R}^{m\times m}$ eigenvectors of $AA^\top$
   - $V \in \mathbb{R}^{n\times n}$ eigenvectors of $A^\top A$
2. **Decomposition of Symmetric Matrices** Suppose Q is symmetric, then $Q = U\Lambda U^\top$, where U is orthogonal and Λ is diagonal. The eigenvalues of Q are the diagonal elements of Λ. The eigenvectors of Q are the columns of U.
3. **QR decomposition:** For a real square matrix A, there exits an orthogonal Q and an upper triangular R such that $A = QR$. If A is nonsingular, then this factorization is unique.
4. **Cholesky-decomposition** Assume A is a symmetric positive square matrix, then $A = U^\top U = UU^\top$ where U is a unique upper triangular matrix and L is a lower triangular matrix.

1.3.2 Matrix Norms

1. **Frobenius norm:** $\|A\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2} = \sqrt{\text{tr}(A^\top A)} = \sqrt{\sum_{i=1}^r \sigma_i^2}$.
2. **Spectral norm:** $\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2 = \sigma_1(A)$.
3. **1-norm**: $\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^m |a_{ij}| = \max_{\|x\|_1=1} \|Ax\|_\infty$.
4. **Operator norm:** $\|A\|_p = \max_{\|x\|_p=1} \|Ax\|_p$.
5. **Infinity norm**: $\|A\|_\infty = \max_{1\le i\le m} \sum_{j=1}^n |a_{ij}| = \max_{\|x\|_\infty=1} \|Ax\|_1$.
6. **Nuclear norm:** $\|A\|_* = \sum_{i=1}^r \sigma_i(A) = \sum_{i=1}^n \sigma_i(A^\top)$.
7. **Induced norm:** $\|A\|_{p\to q} = \max_{\|x\|_p=1} \|Ax\|_q$.

1.3.3 Derivatives of Matrices, Vectors

1. $\frac{d}{dX} \text{tr}\, AX = A^T$, $\frac{d}{dA} \text{tr}\, AX = X^T$
2. $\frac{d}{dx} x^\top a = \frac{d}{dx} a^\top x = a$
3. $\frac{d}{dx} x^\top Ax = (A + A^\top)x$
4. $\frac{d}{dX} a^\top X^\top b = \frac{1}{2}(ba^\top + ab^\top)$ when $X \in \mathcal{S}^n$
5. $\frac{d}{dX} a^\top Xb = ab^\top$, $\frac{d}{dX} a^\top X^\top b = ba^\top$
6. $\frac{d}{dX} b^\top X^\top Xc = X(bc^\top + cb^\top)$
7. $\frac{d}{dX} \text{tr}\,[F(X)] = f(X)^\top$ where $f = F'$
8. $\frac{d}{dX} \log\det X = (X^{-1})^\top$
9. $\frac{d}{dX} \det X = \det X \cdot (X^{-1})^\top$
10. $d(A^{-1}) = -A^{-1}dAA^{-1}$
11. $\frac{d}{dX} \sum eig(X) = \frac{d}{dX} \text{tr}\, X = I$
12. $\frac{d}{dX} \prod eig(X) = \frac{d}{dX} \det(X) = det(X)X^{-\top}$
13. **Norm Derivatives:**
    - $\frac{\partial}{\partial x}\|x - a\|_2 = \frac{x-a}{\|x-a\|_2}$
    - $\frac{\partial}{\partial x}\left(\frac{x-a}{\|x-a\|_2}\right) = \frac{I}{\|x-a\|_2} - \frac{(x-a)(x-a)^T}{\|x-a\|_2^3}$
    - $\frac{\partial}{\partial x}\|x\|_2^2 = \frac{\partial}{\partial x}(x^T x) = 2x$
    - **Frobenius norm:** $\frac{\partial}{\partial X}\|X\|_F^2 = \frac{\partial}{\partial X}\text{Tr}(XX^H) = 2X$

1.4 Reasoning about Unboundedness for Quadratic Forms (Duality)

- $\inf_x\ x^T Qx$ is unbounded below $\iff$ Q is not positive semidefinite $(Q \not\succeq 0)$
- $\sup_x\ x^T Qx$ is unbounded above $\iff$ Q is not negative semidefinite $(Q \not\preceq 0)$
- $\inf_x\ \frac{1}{2}x^T Qx + c^T x$ is unbounded below $\iff$ $Q \not\succeq 0$ or $Q \succeq 0$ and $c \notin \text{range}(Q)$
- $\sup_x; -\frac{1}{2}x^T Qx + c^T x$ is unbounded above $\iff$ $Q \not\preceq 0$ or $Q \preceq 0$ and $c \notin \text{range}(Q)$
- If $Q \succ 0$ (positive definite), then $\inf_x \frac{1}{2}x^T Qx + c^T x$ is finite and achieved at $x^* = -Q^{-1}c$
- If $Q \succeq 0$ (positive semidefinite) and $c \in \text{range}(Q)$, $\inf_x \frac{1}{2}x^T Qx + c^T x$ is finite and achieved at x^* solving $Qx^* + c = 0$
- If Q is indefinite (has both positive and negative eigenvalues), $x^T Qx$ is unbounded both above and below
- $\inf_{x\ge 0} x^T Qx$ is unbounded below if $\exists d \ge 0, d \ne 0$ such that $d^T Qd < 0$
- For $Q \succ 0$, $\sup_x -\frac{1}{2}x^T Qx + c^T x$ is finite and achieved at $x^* = Q^{-1}c$
- For $Q \prec 0$, $\sup_x \frac{1}{2}x^T Qx + c^T x$ is finite and achieved at $x^* = -Q^{-1}c$

1.5 Miscellaneous

1. **Optimization facts:** if min. (max) problem infeasible, define optimal value as $\infty(-\infty)$
2. **Decomposition of any vector:**
   - Orthogonal decomposition of any vector c w.r.t $Ax = b$: $c = A^\top\lambda + c_0$, $A^\top c_0 = 0$
   - Parallel-orthogonal decomposition of any vector c w.r.t vector a: $c = \lambda a + c_0$, $a^\top c_0 = 0$
   - $x = x^+ + x^-$, $x^+ = \max(0, x_i)$, $x^- = \max(0, -x_i)$
3. **Invertible Matrix:**
   - $AA^{-1} = A^{-1}A = I_n$ (unique inverse exists)
   - $\det(A) \ne 0$ (nonzero determinant)
   - Columns and rows are linearly independent
   - Full rank: $\text{rank}(A) = n$
   - The only solution to $A\mathbf{x} = 0$ is $\mathbf{x} = 0$
   - For any $\mathbf{b}$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution
   - $I + yy^T - ss^T/s^T s$ is invertible if and only if $y^\top s \ne 0$
   - $det(I + yy^\top) = 1 + y^\top y$
   - $det(A - uv^\top) = det(A)(1 - v^\top A^{-1}u)$
   - $(I + yy^\top)^{-1} = I - (yy^\top)/(1 + y^\top y)$
   - $eig(A - \lambda I) = eig(A) - \lambda$
   - $\sum_i v_i x_i^2 = x^T diag(v)x$
4. **Symmetric Matrix:**
   - $A = A^T$
   - All eigenvalues of A are real
   - Eigenvectors corresponding to distinct eigenvalues are orthogonal
   - **Spectral Theorem:** A can be diagonalized by an orthogonal matrix: $A = Q\Lambda Q^\top$, where Q is orthogonal ($Q^\top Q = I$), and Λ is diagonal with real entries (the eigenvalues of A)
   - A has an orthonormal basis of eigenvectors
   - The singular values of A are $|\lambda_i(A)|$ (the absolute values of its eigenvalues)
   - $xx^\top$ is always psd
5. **Determinant Properties:**
   - $\det(S) > 0$ if S is pd
   - $\det(A) = \prod_i \lambda_i$, where $\lambda_i = \text{eig}(A)$
   - $\det(cA) = c^n \det(A)$, $\quad A \in \mathbb{R}^{n\times n}$
   - $\det(A^T) = \det(A)$
   - $\det(AB) = \det(A)\det(B)$
   - $\det(A^{-1}) = \frac{1}{\det(A)}$
   - $\det(A^n) = [\det(A)]^n$
   - $\det(I + uv^T) = 1 + u^T v$
   - For $n = 2$: $\det(I + A) = 1 + \det(A) + \text{Tr}(A)$
   - For $n = 3$: $\det(I+A) = 1+\det(A)+\text{Tr}(A)+\frac{1}{2}[\text{Tr}(A)^2 - \text{Tr}(A^2)]$
6. **Range and Null Space:**
   - Range: $R(A) = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\}$
   - Null space: $N(A) = \{\mathbf{x} : A\mathbf{x} = 0\}$
   - Orthogonality: $R(A)^\perp = N(A^\top)$, $N(A) = (R(A^\top))^\perp$
7. **Symmetric Matrix:**
   - If A is symmetric, then $\sigma_i(A) = |\lambda_i(A)|$, where $\lambda_i(A)$ is the i-th largest eigenvalue.
   - $xx^\top$ is always psd
8. **Vector Taylor Expansion:** $f(\mathbf{y}) \approx f(\mathbf{x}) + \nabla f(\mathbf{x})^\top(\mathbf{y} - \mathbf{x}) + \frac{1}{2}(\mathbf{y} - \mathbf{x})^\top \nabla^2 f(\mathbf{x})(\mathbf{y} - \mathbf{x})$
9. **Integral Form of Taylor's Theorem:**
   - $f(y) = f(x) + \int_0^1 \nabla f(x + t(y - x))^\top (y - x)\,dt$
10. **Matrix Rank:**
    - $\text{rank}(A) = \text{rank}(A^\top) = \text{rank}(AA^\top) = \text{rank}(A^\top A)$
11. **Matrix Inequality:**
    - $P \succeq Q \iff I \succeq P^{-1/2}QP^{-1/2}$
12. **Taylor Series Expansions:**
    - $e^x = \sum_{n=0}^\infty \frac{x^n}{n!}$
    - $\log(1 + x) = \sum_{n=1}^\infty (-1)^{n-1}\frac{x^n}{n}$
13. **Logarithm Bounds:**
    - $\frac{x}{1+x} \le \log(1 + x) \le x \quad$ for all $x > -1$
14. **Fundamental Theorem of Calculus:**
    - If f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)dt$, then $F'(x) = f(x)$.
    - If F is an antiderivative of f on $[a, b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.
15. **Basic Integrals:**
    - $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \ne -1$
    - $\int \frac{1}{x} dx = \ln|x| + C$
    - $\int e^{ax} dx = \frac{1}{a}e^{ax} + C$
    - $\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + C$
    - $\int \cos(ax)dx = \frac{1}{a}\sin(ax) + C$
    - $\int \frac{1}{1+x^2} dx = \arctan(x) + C$
    - $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin(x) + C$
16. **Integral Properties:**
    - Linearity: $\int_a^b [af(x) + bg(x)]\,dx = a\int_a^b f(x)dx + b\int_a^b g(x)dx$
    - Additivity: $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$
    - Reversal of limits: $\int_a^b f(x)dx = -\int_b^a f(x)dx$
    - If $f(x) \ge 0$ on $[a, b]$ then $\int_a^b f(x)dx \ge 0$
    - Absolute value: $|\int_a^b f(x)dx| \le \int_a^b |f(x)|dx$
17. **Mean Value Theorem (MVT):**
    - **Differentiation:** If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that:
    $$f'(c) = \frac{f(b) - f(a)}{b - a}$$
    - **Rolle's Theorem:** If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.
    - **Mean Value Theorem for Integrals:** If f is continuous on $[a, b]$, then there exists $c \in [a, b]$ such that:
    $$\int_a^b f(x)dx = f(c)(b - a)$$
18. **Limit Definition of Derivative:**
    - For $f : \mathbb{R} \to \mathbb{R}$, the derivative at x is
    $$f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$$
    - For $f : \mathbb{R}^n \to \mathbb{R}$, the partial derivative with respect to x_i at x is
    $$\frac{\partial f}{\partial x_i}(x) = \lim_{h\to 0} \frac{f(x + he_i) - f(x)}{h}$$
19. **Limit Definition of Hessian (Second Derivatives):**
    - For $f : \mathbb{R}^n \to \mathbb{R}$, the second partial derivative is
    $$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \lim_{h\to 0} \frac{1}{h}\left(\frac{\partial f}{\partial x_i}(x + he_j) - \frac{\partial f}{\partial x_i}(x)\right)$$
    - Alternatively,
    $$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \lim_{h\to 0} \frac{f(x + he_i + he_j) - f(x + he_i) - f(x + he_j}{h^2}$$
    where e_i and e_j are standard basis vectors.
20. **Limits:**
    - $\lim_{x\to a} f(x)$: The value $f(x)$ approaches as x approaches a
    - $\lim_{x\to 0} \frac{\sin x}{x} = 1$
    - $\lim_{x\to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$
    - $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$
    - $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$
    - $\lim_{x\to 0} (1 + x)^{1/x} = e$
    - $\lim_{x\to 0^+} x^a \ln x = 0$ for $a > 0$
21. **Lipschitz Condition:**
    - A function $f : D \to \mathbb{R}$ is **Lipschitz continuous** on D if there exists $L \ge 0$ such that for all $x, y \in D$:
    $$\|f(x) - f(y)\| \le L\|x - y\|$$
    - L is called the **Lipschitz constant**.

2 Convex Sets

Definition 1 Let $S \subseteq \mathbb{R}^n$ be a set. We say that
1. S is affine if $\alpha x + (1 - \alpha)y \in S$ whenever $x, y \in S$ and $\alpha \in \mathbb{R}$;
2. S is convex if $\alpha x + (1 - \alpha)y \in S$ whenever $x, y \in S$ and $\alpha \in [0, 1]$.

Given $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, the vector $z = \alpha x + (1 - \alpha)y$ is called an *affine combination* of x and y. If $\alpha \in [0, 1]$, then z is called a *convex combination* of x and y.

Geometrically, when x and y are distinct points in $\mathbb{R}^n$, the set $L = \{z \in \mathbb{R}^n : z = \alpha x + (1 - \alpha)y, \alpha \in \mathbb{R}\}$ of all affine combinations of x and y is the *line* determined by x and y, and the set $S = \{z \in \mathbb{R}^n : z = \alpha x + (1 - \alpha)y, \alpha \in [0, 1]\}$ is the *line segment* between x and y. By convention, the empty set $\emptyset$ is affine and hence also convex.

linear $\Rightarrow$ affine $\Rightarrow$ convex.

1. An affine combination of the points $x_1, \ldots, x_k \in \mathbb{R}^n$ is a point of the form $z = \sum_{i=1}^k \alpha_i x_i$, where $\sum_{i=1}^k \alpha_i = 1$.
2. A convex combination of the points $x_1, \ldots, x_k \in \mathbb{R}^n$ is a point of the form $z = \sum_{i=1}^k \alpha_i x_i$, where $\sum_{i=1}^k \alpha_i = 1$ and $\alpha_1, \ldots, \alpha_k \ge 0$.

Proposition 1 Let $S \subseteq \mathbb{R}^n$ be non-empty. The following are equivalent:
(a) S is affine.
(b) Any affine combination of points in S belongs to S.
(c) S is the translation of some linear subspace $V \subseteq \mathbb{R}^n$; i.e., S is of the form $\{x\} + V = \{x + v \in \mathbb{R}^n : v \in V\}$ for some $x \in \mathbb{R}^n$.

Note: Though V is unique, x is not. V is parallel of S pass 0. For n dimensional S, there are $(n + 1)$ types of subspaces: origin, line, plane, ... $\{x : Ax = b\}$ is affine. $\{x : Ax \le b\}$ is convex but not affine.

Proposition 2 Let $S \subseteq \mathbb{R}^n$ be arbitrary. Then, the following are equivalent:
(a) S is convex.
(b) Any convex combination of points in S belongs to S.

Definition A set $K \subseteq \mathbb{R}^n$ is called a *cone* if $\{\alpha x : \alpha > 0\} \subseteq K$ whenever $x \in K$. If K is also convex, then K is called a *convex cone*.

A cone need not be convex. Eg: two lines.

Definition 2 Let $S \subseteq \mathbb{R}^n$ be arbitrary.
1. The affine hull of S, aff(S), is the intersection of all affine subspaces containing S. aff(S) is the smallest affine subspace that contains S.
2. The convex hull of S, conv(S), is the intersection of all convex sets containing S. conv(S) is the smallest convex set that contains S.

Proposition 3 Let $S \subseteq \mathbb{R}^n$ be arbitrary. Then, the following hold:
(a) aff(S) is the set of all affine combinations of points in S.
(b) conv(S) is the set of all convex combinations of points in S.

Definition 3 Let $S \subseteq \mathbb{R}^n$ be arbitrary. The dimension of S, denoted by $\dim(S)$, is the dimension of the affine hull of S.

Given a non-empty set $S \subseteq \mathbb{R}^n$, we always have $0 \le \dim(S) \le n$.

Example 2 (Dimension of a Set) Consider the two-point set $S = \{(1, 1), (3, 2)\} \subseteq \mathbb{R}^2$. By Proposition 3(a), we have aff$(S) = \{\alpha(1, 1) + (1 - \alpha)(3, 2) : \alpha \in \mathbb{R}\} \subseteq \mathbb{R}^2$. It is easy to verify that aff$(S) = \{(0, 1/2)\} + V$, where $V = \{t(1, 1/2) : t \in \mathbb{R}\}$ is the linear subspace generated by the vector $(1, 1/2)$. Hence, we have $\dim(S) = \dim(V) = 1$.

2.1 Convexity-Preserving Operations

2.1.1 Set Operations

Intersection of two convex sets is always convex.

2.1.2 Affine Functions

We say that a map $A : \mathbb{R}^n \to \mathbb{R}^m$ is affine if $A(\alpha x_1 + (1 - \alpha)x_2) = \alpha A(x_1) + (1 - \alpha)A(x_2)$ for all $x_1, x_2 \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. It can be shown that A is affine iff there exist $A_0 \in \mathbb{R}^{m\times n}$ and $y_0 \in \mathbb{R}^m$ such that $A(x) = A_0 x + y_0$ for all $x \in \mathbb{R}^n$.

Proposition 4 Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be an affine mapping and $S \subseteq \mathbb{R}^n$ be a convex set. Then, the image $A(S) = \{A(x) \in \mathbb{R}^m : x \in S\}$ is convex. Conversely, if $T \subseteq \mathbb{R}^m$ is a convex set, then the inverse image $A^{-1}(T) = \{x \in \mathbb{R}^n : A(x) \in T\}$ is convex.

(a) Rotation: $\mathbb{R}^n \ni x \mapsto A(x) = Ux \in \mathbb{R}^n$
$U \in \mathbb{R}^{n\times n}$ orthogonal matrix (i.e., $U^\top U = UU^\top = I$)

For $n = 2$: $U = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

(b) Projection: $\mathbb{R}^n \ni x \mapsto A(x) = Px \in \mathbb{R}^n$
$P \in \mathbb{R}^{n\times n}$ projection matrix (i.e., $P^2 = P$)
orthogonal projection (i.e., $P^2 = P$, $P = P^\top$)

eg. $P = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $Px = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix}$, $P' = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}ee^\top$.

2.1.3 Perspective Functions

Define the *perspective function* $P : \mathbb{R}^n \times \mathbb{R}_{++} \to \mathbb{R}^n$ by $P(x,t) = x/t$.

Proposition 5 Let $P : \mathbb{R}^n \times \mathbb{R}_{++} \to \mathbb{R}^n$ be the perspective function and $S \subseteq \mathbb{R}^n \times \mathbb{R}_{++}$ be a convex set. Then, the image $P(S) = \{x/t \in \mathbb{R}^n : (x,t) \in S\}$ is convex. Conversely, if $T \subseteq \mathbb{R}^n$ is a convex set, then the inverse image $P^{-1}(T) = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}_{++} : x/t \in T\}$ is convex.

Proof For any $x_1 = (\bar{x}_1, t_1) \in \mathbb{R}^n \times \mathbb{R}_{++}$, $x_2 = (\bar{x}_2, t_2) \in \mathbb{R}^n \times \mathbb{R}_{++}$, and $\alpha \in [0,1]$, we have $P(\alpha x_1 + (1-\alpha)x_2) = \frac{\alpha\bar{x}_1 + (1-\alpha)\bar{x}_2}{\alpha t_1 + (1-\alpha)t_2} = \beta P(x_1) + (1-\beta)P(x_2)$,
where $\beta = \frac{\alpha t_1}{\alpha t_1 + (1-\alpha)t_2} \in [0,1]$.
Moreover, as α increases from 0 to 1, β increases from 0 to 1. It follows that $P([x_1, x_2]) = [P(x_1), P(x_2)] \subseteq \mathbb{R}^n$. This completes the proof. □

Corollary 1 Let $A : \mathbb{R}^n \to \mathbb{R}^{m+1}$ be the affine map given by

$A(x) = \begin{bmatrix} Q \\ c^\top \end{bmatrix} x + \begin{bmatrix} u \\ d \end{bmatrix}$, where $Q \in \mathbb{R}^{m\times n}$, $c \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $d \in \mathbb{R}$.

Let $D = \{x \in \mathbb{R}^n : c^\top x + d > 0\}$. Define the *linear-fractional map* $f : D \to \mathbb{R}^m$ by $f = P \circ A$, where $P : \mathbb{R}^m \times \mathbb{R}_{++} \to \mathbb{R}^m$ is the perspective function. If $S \subseteq D$ is convex, then the image $f(S)$ is convex. Conversely, if $T \subseteq \mathbb{R}^m$ is convex, then the inverse image $f^{-1}(T)$ is convex.

2.2 Topological Properties

1. **Interior**: $\text{int}(S) = \{x \in S : B(x,\epsilon) \subseteq S \text{ for some } \epsilon > 0\}$.
2. S is **open** if $S = \text{int}(S)$. Eg. $\emptyset, \mathbb{R}^n$ are open.
3. S is **closed** if $\mathbb{R}^n \setminus S$ is open. Eg. $\emptyset, \mathbb{R}^n$ are closed.
4. A set $S \subseteq \mathbb{R}^n$ is **compact** if it is closed and bounded.

Facts: 1. The intersection of any family of closed sets is closed.
2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and $c \in \mathbb{R}$ be a constant. Then $S = \{x \in \mathbb{R}^n : f(x) \le c\}$ is closed.
Eg. $H^-(s,c) = \{x \in \mathbb{R}^n : s^\top x \le c\}$; $S = \{x \in \mathbb{R}^n : Ax \le b\}$
3. S is closed iff $\forall$ convergent sequence $\{x_n\}$ in S ($x_n \in S$, $x_n \to x$), the limit is in S ($x \in S$).

Definition 4 Let $S \subseteq \mathbb{R}^n$ be arbitrary. We say that $x \in S$ belongs to the relative interior of S, denoted by $x \in \text{rel int}(S)$, if there exists an $\epsilon > 0$ such that $B(x,\epsilon) \cap \text{aff}(S) \subseteq S$. The relative boundary of S, denoted by $\text{rel bd}(S)$, is defined by $\text{rel bd}(S) = \text{cl}(S) \setminus \text{rel int}(S)$.

Theorem 1 Let $S \subseteq \mathbb{R}^n$ non-empty and convex, rel int(S) is non-empty.

Proposition 6 $S \subseteq \mathbb{R}^n$ be non-empty and convex. $\forall x \in \text{cl}(S)$ and $x' \in \text{rel int}(S)$, $(x, x'] = \{\alpha x + (1-\alpha)x' \in \mathbb{R}^n : \alpha \in [0,1)\} \subseteq \text{rel int}(S)$.

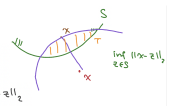
Weierstrass Theorem If $S \subseteq \mathbb{R}^n$ is a non-empty compact set and $f : S \to \mathbb{R}$ is a continuous function, then f attains its maximum and minimum on S.

2.3 Projection onto Closed Convex Sets

Theorem 2 Let $S \subseteq \mathbb{R}^n$ be non-empty, closed, and convex. Then, for every $x \in \mathbb{R}^n$, there exists a unique point $z^* \in S$ that is closest to x.

Proof (Existence)

We may assume that $x \notin S$.
Consider any $x' \in S$ and define
$T = S \cap B(x, \|x - x'\|_2)$



① $\text{argmin}_{z\in S} \|x - z\|_2 = \text{argmin}_{z\in T} \|x - z\|_2$
② T is closed (verify) and bounded ($T \subseteq B(x, \|x - x'\|_2)$)
③ By Weierstrass' theorem, $z^* = \Pi_T(x)$ exists. By ①, $z^* = \Pi_S(x)$

(Uniqueness) Let $\mu^* = \|x - z^*\|_2$ and suppose that $z_1, z_2 \in S$ are such that $\mu^* = \|x - z_1\|_2 = \|x - z_2\|_2$. Consider the point $\bar{z} = \frac{1}{2}(z_1 + z_2)$. By Pythagoras' theorem, we have $\|\bar{z} - x\|_2^2 = (\mu^*)^2 - \frac{1}{4}\|z_1 - z_2\|_2^2$. In particular, if $z_1 \ne z_2$, then $\|\bar{z} - x\|_2^2 < (\mu^*)^2$, which is a contradiction. □

Point z^* is the *projection* of x on S: $\Pi_S(x) = \arg\min_{z\in S} \|x - z\|_2^2$.

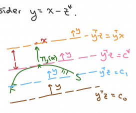
Theorem 3 Let $S \subseteq \mathbb{R}^n$ non-empty, closed, convex. Given any $x \in \mathbb{R}^n$, we have $z^* = \Pi_S(x)$ iff $z^* \in S$ and $(z - z^*)^\top(x - z^*) \le 0$ for all $z \in S$.

Proof Let $z^* = \Pi_S(x)$ and $z \in S$. Consider points of the form $z(\alpha) = \alpha z + (1-\alpha)z^*$, where $\alpha \in [0,1]$. By convexity, we have $z(\alpha) \in S$. Moreover, we have $\|z^* - x\|_2 \le \|z(\alpha) - x\|_2$ for all $\alpha \in [0,1]$. On the other hand, note that $\|z(\alpha) - x\|_2^2 = \|z^* + \alpha(z - z^*) - x\|_2^2 = \|z^* - x\|_2^2 + 2\alpha(z - z^*)^\top(z^* - x) + \alpha^2\|z - z^*\|_2^2$. Thus, we see that $\|z(\alpha) - x\|_2^2 \ge \|z^* - x\|_2^2$ for all $\alpha \in [0,1]$ iff $(z - z^*)^\top(z^* - x) \ge 0$.
Conversely, suppose that for some $z' \in S$, we have $(z - z')^\top(x - z') \le 0$ for all $z \in S$. Upon setting $z = \Pi_S(x)$, we have $(\Pi_S(x) - z')^\top(x - z') \le 0$. (1) On the other hand, by our argument above, the point $\Pi_S(x)$ satisfies $(z' - \Pi_S(x))^\top(x - \Pi_S(x)) \le 0$. (2) Upon adding (1) and (2), we obtain $(\Pi_S(x) - z')^\top(\Pi_S(x) - z') = \|\Pi_S(x) - z'\|_2^2 \le 0$, which is possible only when $z' = \Pi_S(x)$. □

2.4 Separation Theorems

Theorem 4 (Point-Set Separation) Let $S \subseteq \mathbb{R}^n$ be non-empty, closed, and convex and $x \notin S$. Then, $\exists y \in \mathbb{R}^n$ s.t. $\max_{z\in S} y^\top z < y^\top x$.

Proof: By the projection theorem,
$z^* = \Pi_S(x)$ exists and is unique. Consider $y = x - z^*$.
Note that $y \ne 0$ since $x \notin S$. Moreover,
$\forall z \in S$: $(z - z^*)^\top \underbrace{(x - z^*)}_{y} \le 0$
$\Rightarrow y^\top z \le y^\top z^* = y^\top x + y^\top \underbrace{(z^* - x)}_{-y} = y^\top x - \|y\|_2^2$
$\Rightarrow \max_{z\in S} y^\top z \le y^\top x - \|y\|_2^2 < y^\top x$



Theorem 5 A closed convex set $S \subseteq \mathbb{R}^n$ is the intersection of all the halfspaces containing S, i.e., $S = \bigcap_{H \text{ is halfspace}; H \supseteq S} H$.

Proof We may assume that $\emptyset \subset S \subseteq \mathbb{R}^n$, for otherwise the theorem is trivial. Let $x \in \mathbb{R}^n \setminus S$ be arbitrary. Then, by Theorem 4, there exist $y \in \mathbb{R}^n$ and $c = \max_{z\in S} y^\top z \in \mathbb{R}$ such that the halfspace $H^-(y,c) = \{z \in \mathbb{R}^n : y^\top z \le c\}$ contains S but not x. It follows that the intersection of all the halfspaces containing S is precisely S itself. □

Theorem 6 (Set-Set Separation) Let $S_1, S_2 \subseteq \mathbb{R}^n$ be non-empty, closed, and convex with $S_1 \cap S_2 = \emptyset$ and S_2 bounded. Then, there exists a $y \in \mathbb{R}^n$ such that $\max_{z\in S_1} y^\top z < \min_{u\in S_2} y^\top u$.

Proof First, note that the set $S_1 - S_2 = \{z - u \in \mathbb{R} : z \in S_1, u \in S_2\}$ is non-empty and convex. Moreover, we claim that it is closed. To see this, let $x_1, x_2, \ldots$ be a sequence in $S_1 - S_2$ such that $x_k \to x$. We need to show that $x \in S_1 - S_2$. Since $x_k \in S_1 - S_2$, there exist $z_k \in S_1$ and $u_k \in S_2$ such that $x_k = z_k - u_k$ for $k = 1, 2, \ldots$. Since S_2 is compact, there exists a subsequence $\{u_{k_i}\}$ such that $u_{k_i} \to u \in S_2$. Since $x_{k_i} \to x$, we conclude that $z_{k_i} \to x + u$. Since S_1 is closed, we conclude that $x + u \in S_1$. It then follows that $x = (x + u) - u \in S_1 - S_2$, as desired.
We are now in a position to apply Theorem 4 to the non-empty closed convex set $S_1 - S_2$. Indeed, since $S_1 \cap S_2 = \emptyset$, we see that $\mathbf{0} \notin S_1 - S_2$. By Theorem 4, there exist $y \in \mathbb{R}^n$, $z^* \in S_1$, and $u^* \in S_2$ such that $y^\top(z^* - u^*) = \max_{v\in S_1 - S_2} y^\top v < 0$. Since S_2 is compact, we have $y^\top u^* = \min_{u\in S_2} y^\top u$. This implies that $y^\top z^* = \max_{z\in S_1} y^\top z$. Hence, we obtain $\max_{z\in S_1} y^\top z < \min_{u\in S_2} y^\top u$, as desired. □

3 Convex Functions

3.1 Basic Definitions and Properties

Definition 5 Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function that is not identically $+\infty$.

1. We say f is convex if $f(\alpha x_1 + (1-\alpha)x_2) \le \alpha f(x_1) + (1-\alpha)f(x_2)$ for all $x_1, x_2 \in \mathbb{R}^n$ and $\alpha \in [0,1]$; f is concave if $-f$ is convex.
2. The epigraph of f is the set $\text{epi}(f) = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : f(x) \le t\}$.
3. The effective domain of f is the set $\text{dom}(f) = \{x \in \mathbb{R}^n : f(x) < +\infty\}$.
4. The indicator of S is the function $\mathbb{I}_S(x) = \begin{cases} 0 & \text{if } x \in S \\ +\infty & \text{otherwise} \end{cases}$

We have $\inf_{x\in S} f(x) \iff \inf_{x\in\mathbb{R}^n} \{f(x) + \mathbb{I}_S(x)\}$

Proposition 7 Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. Then, f is convex (as a function) iff epi(f) is convex (as a set).
Note: Also, set S is convex iff $\mathbb{I}_S(x)$ is convex.

Corollary 2 (Jensen's Inequality) Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be as in Definition 5. Then, f is convex iff $f\left(\sum_{i=1}^k \alpha_i x_i\right) \le \sum_{i=1}^k \alpha_i f(x_i)$ for any $x_1, \ldots, x_k \in \mathbb{R}^n$ and $\alpha_1, \ldots, \alpha_k \in [0,1]$ such that $\sum_{i=1}^k \alpha_i = 1$.
Note: The epigraph epi(f) of f is closely related to, but not the same as, the t-level set $L_t(f)$ of f, where $L_t(f) = \{x \in \mathbb{R}^n : f(x) \le t\}$ and $t \in \mathbb{R}$ is arbitrary. Even if $L_t(f)$ is convex for all $t \in \mathbb{R}$, the function f may not be convex. E.g., the function $x \mapsto x^3$. A function whose domain is convex and whose t-level sets are convex for all $t \in \mathbb{R}$ is called a *quasi-convex function*.

3.2 Conjugate Function

Theorem 7 Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function such that epi(f) is closed. Then, f can be represented as the pointwise supremum of all affine functions $h : \mathbb{R}^n \to \mathbb{R}$ satisfying $h \le f$.
Given a convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, consider the set $S_f = \{(y,c) \in \mathbb{R}^n \times \mathbb{R} : y^\top x - c \le f(x) \forall x \in \mathbb{R}^n\}$, which consists of the coefficients of those affine functions $h : \mathbb{R}^n \to \mathbb{R}$ satisfying $h \le f$. Clearly, we have $y^\top x - c \le f(x)$ for all $x \in \mathbb{R}^n$ iff $\sup_{x\in\mathbb{R}^n}\{y^\top x - f(x)\} \le c$. This shows that S_f is the epigraph of the function $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ given by $f^*(y) = \sup_{x\in\mathbb{R}^n}\{y^\top x - f(x)\}$. Moreover, observe that S_f is closed and convex, implies f^* is convex. f^* is called the *conjugate* of f.

3.3 Convexity-Preserving Transformations

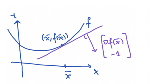
Theorem 8 The following hold:

(a) (Non-Negative Combinations) Let $f_1, \ldots, f_m$ be convex functions satisfying $\bigcap_{i=1}^m \text{dom}(f_i) \ne \emptyset$. Then, for any $\alpha_1, \ldots, \alpha_m \ge 0$, $f(x) = \sum_{i=1}^m \alpha_i f_i(x)$ is convex.
(b) (Pointwise Supremum) Let I be an index set and $\{f_i\}_{i\in I}$, where $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ for all $i \in I$, be a family of convex functions. Define the *pointwise supremum* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ of $\{f_i\}_{i\in I}$ by $f(x) = \sup_{i\in I} f_i(x)$. Suppose that dom$(f) \ne \emptyset$. Then f is convex.
(c) (Affine Composition) Let $g : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ be a convex function and $A : \mathbb{R}^m \to \mathbb{R}^n$ be an affine mapping. Suppose range$(A) \cap \text{dom}(g) \ne \emptyset$. Then $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined by $f(x) = g(A(x))$ is convex.
(d) (Composition with an Increasing Convex Function) Let $g : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ and $h : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be convex functions that are not identically $+\infty$. Suppose that h is increasing on dom(h). Define $f(x) = h(g(x))$, with the convention that $h(+\infty) = +\infty$. Suppose that dom$(f) \ne \emptyset$. Then, f is convex.
(e) (Restriction on Lines) Given a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ that is not identically $+\infty$, a point $x_0 \in \mathbb{R}^n$, and a direction $h \in \mathbb{R}^n$, define the function $\tilde{f}_{x_0,h} : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ by $\tilde{f}_{x_0,h}(t) = f(x_0 + th)$. Then, f is convex iff $\tilde{f}_{x_0,h}$ is convex for any $x_0 \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$.

3.4 Differentiable Convex Functions

Theorem 9 Let $f : \Omega \to \mathbb{R}$ be a differentiable function on the open set $\Omega \subseteq \mathbb{R}^n$ and $S \subseteq \Omega$ be a convex set. Then, f is convex on S iff $f(x) \ge f(\bar{x}) + (\nabla f(\bar{x}))^\top(x - \bar{x})$ for all $x, \bar{x} \in S$.
Geometric interpretation: the epigraph of f is supported by its tangent hyperplane at every $(\bar{x}, f(\bar{x})) \in \mathbb{R}^n \times \mathbb{R}$.

Proof Suppose that f is convex on S. Let $x, \bar{x} \in S$ and $\alpha \in (0,1)$. Then, $f(x) \ge \frac{f(\alpha x + (1-\alpha)\bar{x}) - (1-\alpha)f(\bar{x})}{\alpha} = f(\bar{x}) + \frac{f(\bar{x} + \alpha(x - \bar{x})) - f(\bar{x})}{\alpha}$.
Now, recall that $\lim_{\alpha\downarrow 0} \frac{f(\bar{x} + \alpha(x - \bar{x})) - f(\bar{x})}{\alpha}$ is the directional derivative of f at $\bar{x}$ in the direction $x - \bar{x}$ and is equal to $(\nabla f(\bar{x}))^\top(x - \bar{x})$. Hence, upon letting $\alpha \downarrow 0$ in (4), we have $f(x) \ge f(\bar{x}) + (\nabla f(\bar{x}))^\top(x - \bar{x})$.
Conversely, let $x_1, x_2 \in S$ and $\alpha \in (0,1)$. Then, we have $\bar{x} = \alpha x_1 + (1-\alpha)x_2 \in S$, which implies that $f(x_1) \ge f(\bar{x}) + (1-\alpha)(\nabla f(\bar{x}))^\top(x_1 - x_2)$; $f(x_2) \ge f(\bar{x}) + \alpha(\nabla f(\bar{x}))^\top(x_2 - x_1)$.
Obtain the result by multiplying first by α and second by $1 - \alpha$ and sum. □



$t = f(\bar{x}) + \nabla f(\bar{x})^\top(x - \bar{x})$
$\Updownarrow$
$\underbrace{\begin{bmatrix} \nabla f(\bar{x}) \\ -1 \end{bmatrix}^\top}_{s} \underbrace{\begin{bmatrix} x \\ t \end{bmatrix}}_{z} + \underbrace{f(\bar{x}) - \nabla f(\bar{x})^\top \bar{x}}_{c} = 0$
$\Leftrightarrow s^\top z = -c$

Theorem 10 Let $f : S \to \mathbb{R}$ be twice continuously differentiable on the open convex set $S \subseteq \mathbb{R}^n$. Then, f is convex on S iff $\nabla^2 f(x) \succeq 0$ for all $x \in S$.

Proof Suppose that $\nabla^2 f(x) \succeq 0$ for all $x \in S$. Let $x_1, x_2 \in S$. By Taylor's theorem, there exists an $\bar{x} \in [x_1, x_2] \subseteq S$ s.t. $f(x_2) = f(x_1) + (\nabla f(x_1))^\top(x_2 - x_1) + \frac{1}{2}(x_2 - x_1)^\top\nabla^2 f(\bar{x})(x_2 - x_1)$.[7]
Since $\nabla^2 f(x) \succeq 0$, we have $(x_2 - x_1)^\top\nabla^2 f(\bar{x})(x_2 - x_1) \ge 0$. Upon substituting this inequality and invoking Theorem 9, f is convex on S.
Conversely, suppose that $\nabla^2 f(\bar{x}) \not\succeq 0$ for some $\bar{x} \in S$. Then, there exists a $v \in \mathbb{R}^n$ such that $v^\top\nabla^2 f(\bar{x})v < 0$. Since S is open and $\nabla^2 f$ is continuous, there exists an $\epsilon > 0$ such that $\bar{x}' = \bar{x} + \epsilon v \in S$ and $v^\top\nabla^2 f(\bar{x} + \alpha(\bar{x}' - \bar{x}))v < 0$ for all $\alpha \in [0,1]$. Hence, by taking $x_1 = \bar{x}$ and $x_2 = \bar{x}'$ in [7], $f(x') < f(\bar{x}) + (\nabla f(\bar{x}))^\top(x' - \bar{x})$. Hence, by Theorem 9, f is not convex on S. □

To see why S must be open in Theorem 10, consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = x^2 - y^2$, is convex on the set $S = \mathbb{R} \times \{0\}$. But Hessian $\nabla^2 f(x,y) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ for $(x,y) \in \mathbb{R}^2$, is nowhere psd.

3.5 Examples of Convex Functions

- Let $f : \mathbb{R}^n \times \mathcal{S}^n_{++} \to \mathbb{R}$ be given by $f(x,Y) = x^\top Y^{-1}x$.
epi$(f) = \{(x,Y,r) \in \mathbb{R}^n \times \mathcal{S}^n_{++} \times \mathbb{R} : Y \succ 0, x^\top Y^{-1}x \le r\}$
$= \left\{(x,Y,r) \in \mathbb{R}^n \times \mathcal{S}^n_{++} \times \mathbb{R} : \begin{bmatrix} Y & x \\ x^\top & r \end{bmatrix} \succeq 0, Y \succ 0\right\}$, where the last equality follows from the Schur complement. This shows that epi(f) is a convex set, which implies that f is convex on $\mathbb{R}^n \times \mathcal{S}^n_{++}$.
- Let $f : \mathbb{R}^{m\times n} \to \mathbb{R}_+$ be given by $f(X) = \|X\|_2$, where $\|\cdot\|_2$ denotes the spectral norm or largest singular value of the $m \times n$ matrix X. $f(X) = \sup\{u^\top Xv : \|u\|_2 = 1, \|v\|_2 = 1\}$. This shows that f is a pointwise supremum of a family of linear functions of X. Hence, f is convex.
- Let $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}_+$ be a norm on $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}_+$ be given by $f(x) = \|x\|^p$, where $p \ge 1$. Then, for any $x \in \mathbb{R}^n$, we have $f(x) = g(\|x\|)$, where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is given by $g(z) = z^p$.
- Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(x) = \log\left(\sum_{i=1}^n \exp(x_i)\right)$. $\frac{\partial^2 f}{\partial x_i \partial x_j} = \begin{cases} \frac{\exp(x_i)}{\sum_{i=1}^n \exp(x_i)} - \frac{\exp(2x_i)}{\left(\sum_{i=1}^n \exp(x_i)\right)^2} & \text{if } i = j, \\ -\frac{\exp(x_i + x_j)}{\left(\sum_{i=1}^n \exp(x_i)\right)^2} & \text{if } i \ne j. \end{cases}$
$\nabla^2 f(x) = \frac{1}{(e^\top z)^2}\left((e^\top z)\,\text{diag}(z) - zz^\top\right)$, $z = (e^{x_1}, \ldots, e^{x_n})$.
Now, for any $v \in \mathbb{R}^n$, we have $v^\top\nabla^2 f(x)v$
$= \frac{1}{(e^\top z)^2}\left[\left(\sum_{i=1}^n z_i\right)\left(\sum_{i=1}^n z_i v_i^2\right) - \left(\sum_{i=1}^n z_i v_i\right)^2\right]$
$= \frac{1}{(e^\top z)^2}\left[\left(\sum_{i=1}^n (\sqrt{z_i})^2\right)\left(\sum_{i=1}^n (\sqrt{z_i}v_i)^2\right) - \left(\sum_{i=1}^n (\sqrt{z_i})(\sqrt{z_i}v_i)\right)^2\right]$
≥ 0 by the Cauchy–Schwarz inequality. Hence, f is convex.
- Suppose $p \in (0,1)$. $f : \mathbb{R}^n_{++} \to \mathbb{R}$ given by $f(x) = \left(\sum_{i=1}^n x_i^p\right)^{1/p}$.
$\frac{\partial^2 f}{\partial x_i \partial x_j} = \begin{cases} (1-p)f(x)^{-2}\left[-\left(\sum_{i=1}^n x_i^p\right)x_i^{p-2} + x_i^{2(p-1)}\right] & i = j, \\ (1-p)f(x)^{-2}x_i^{p-1}x_j^{p-1} & i \ne j. \end{cases}$
$\nabla^2 f(x) = (1-p)f(x)^{-2}\left[-\left(\sum_{i=1}^n x_i^p\right)\text{diag}(x_1^{p-2}, \ldots, x_n^{p-2}) + zz^\top\right]$
where $z_i = x_i^{p-1}$ for $i = 1, \ldots, n$.
Now, for any $v \in \mathbb{R}^n$, we have $v^\top\nabla^2 f(x)v$
$= (1-p)f(x)^{-2}\left[-\left(\sum_{i=1}^n x_i^p\right)\left(\sum_{i=1}^n v_i^2 x_i^{p-2}\right) + \left(\sum_{i=1}^n v_i x_i^{p-1}\right)^2\right]$
≤ 0, since $-\left(\sum_{i=1}^n x_i^p\right)\left(\sum_{i=1}^n v_i^2 x_i^{p-2}\right) + \left(\sum_{i=1}^n v_i x_i^{p-1}\right)^2 \le 0$.
by the Cauchy–Schwarz inequality. It follows that f is concave on $\mathbb{R}^n_{++}$.
- Let $f : \mathcal{S}^n_{++} \to \mathbb{R}$ be given by $f(X) = -\ln\det X$.
$\nabla f(X) = -X^{-1}$; $\nabla^2 f(X) = X^{-1} \otimes X^{-1}$. $\otimes$: Kronecker product.
Since $X^{-1} \succ 0$, $X^{-1} \otimes X^{-1} \succ 0$. It follows that f is convex on $\mathcal{S}^n_{++}$.
Alternatively, we can establish the convexity of f on $\mathcal{S}^n_{++}$ by applying Theorem 8(e). To begin, let $X_0 \in \mathcal{S}^n_{++}$ and $H \in \mathcal{S}^n$. Define the set $D = \{t \in \mathbb{R} : X_0 + tH \succ 0\} = \{t \in \mathbb{R} : \lambda_{\min}(X_0 + tH) > 0\}$. Since $\lambda_{\min}$ is continuous, we see that D is open and convex. Now, consider $\tilde{f}_{X_0,H} : D \to \mathbb{R}$ given by $\tilde{f}_{X_0,H}(t) = f(X_0 + tH)$. For any $t \in D$,
$$\tilde{f}_{X_0,H}(t) = -\ln\det(X_0 + tH)$$
$$= -\ln\det\left(X_0^{1/2}\left(I + tX_0^{-1/2}HX_0^{-1/2}\right)X_0^{1/2}\right)$$
$$= -\left(\sum_{i=1}^n \ln(1 + t\lambda_i) + \ln\det X_0\right)$$
and $\tilde{f}''_{X_0,H}(t) = \sum_{i=1}^n \frac{\lambda_i^2}{(1 + t\lambda_i)^2} \ge 0$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $X_0^{-1/2}HX_0^{-1/2}$. So $\tilde{f}_{X_0,H}$ is convex on D. Together with Theorem 8(e), implies f is convex on $\mathcal{S}^n_{++}$.
- **Boyd:** max eigenvalue of symmetric matrix. $f(X) = \sup\{y^\top Xy : \|y\|_2 = 1\}$.

3.6 Non-Differentiable Convex Functions

Definition 6 Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be as in Definition 5. A vector $s \in \mathbb{R}^n$ is called a *subgradient* of f at $\bar{x}$ if $f(x) \ge f(\bar{x}) + s^\top(x - \bar{x}) \forall x \in \mathbb{R}^n$. The set of vectors s is called subdifferential of f at $\bar{x}$ and denoted by $\partial f(\bar{x})$.

Theorem 11 Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function that is not identically $+\infty$.

1. If $x \in \text{int dom} f$, then $\partial f(x)$ is nonempty and bounded.
2. (Subgradient and Directional Derivative) Let $f'(x,d) = \lim_{t\downarrow 0} \frac{f(x + td) - f(x)}{t}$ be the directional derivative of f at $x \in \mathbb{R}^n$ in the direction $d \in \mathbb{R}^n \setminus \{0\}$, and let $x \in \text{int dom}(f)$. Then, $\partial f(x)$ is a non-empty compact set. Moreover, for any $d \in \mathbb{R}^n$, we have $f'(x,d) = \max_{s\in\partial f(x)} s^\top d$.
3. (Subdifferential of a Differentiable Function) The convex function f is differentiable at $x \in \mathbb{R}^n$ iff the subdifferential $\partial f(x)$ is a singleton, in which case it consists of the gradient of f at x.
4. (Additivity of Subdifferentials) Suppose that $f = f_1 + f_2$, where $f_1 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $f_2 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ are convex functions that are not identically $+\infty$. Furthermore, suppose $\exists x_0 \in \text{dom}(f)$ s.t. f_1 is continuous at x_0. Then, we have $\partial f(x) = \partial f_1(x) + \partial f_2(x) \; \forall x \in \text{dom}(f)$.

Example 5 (A Convex Function with Empty Subdifferential at a Point)
$f(x) = \begin{cases} -\sqrt{1 - \|x\|_2^2} & \text{if } \|x\|_2 \le 1, \\ +\infty & \text{otherwise.} \end{cases}$ It is clear that f is a convex function with dom$(f) = B(0,1)$ and is differentiable on int dom$(f) = \{x \in \mathbb{R}^n : \|x\|_2 < 1\}$. It is also clear that $\partial f(x) = \emptyset$ for all $x \notin \text{dom}(f)$. Now, let $\bar{x} \in \text{bd dom}(f) = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$ be arbitrary. If $s \in \partial f(\bar{x})$,
$f(x) = -\sqrt{1 - \|x\|_2^2} \ge s^\top(x - \bar{x})$ for all $x \in B(0,1)$. (9)
W.l.o.g., we may assume that $\bar{x} = e_1$. For $\alpha \in [-1,1]$, define $x(\alpha) = \alpha e_1 \in B(0,1)$. From (9), we see that $s \in \mathbb{R}^n$ satisfies
$f(x(\alpha)) = -\sqrt{1 - \alpha^2} \ge \alpha s_1 - s_1$ for all $\alpha \in [-1,1]$.
However, this implies that $s_1 \ge \sqrt{(1+\alpha)/(1-\alpha)}$ for all $\alpha \in [-1,1]$, which is impossible. Hence, we have $\partial f(\bar{x}) = \emptyset$ for all $\bar{x} \in \text{bd dom}(f)$.

Example 6 (Subdifferential of the Euclidean Norm)
$f(x) = \|x\|_2$. Note that f is differentiable whenever $x \ne 0$. Hence, by Theorem 11(b) we have $\partial f(x) = \{\nabla f(x)\} = \{x/\|x\|_2\}$ for all $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. Now, recall from Definition 6 that $s \in \mathbb{R}^n$ is a subgradient of f at 0 iff $\|x\|_2 \ge s^\top x$ for $x \in \mathbb{R}^n$. It follows that $\partial f(\mathbf{0}) = B(\mathbf{0}, 1)$.

3.7 Example Problems

1. Let $B_\infty = \{x \in \mathbb{R}^n : -1 \le x_i \le 1 \text{ for } i = 1, \ldots, n\}$. For any $x \in B_\infty$, consider the set $N(x) = \{u \in \mathbb{R}^n : u^\top(y - x) \le 0 \text{ for all } y \in B_\infty\}$. (a) Show that $N(x)$ is a convex cone for any $x \in B_\infty$.
(b) Give an explicit description of $N(x)$.
A: (a) Let $x \in B_\infty$ be fixed. For any $\alpha > 0$ and $u \in N(x)$, $\alpha u^\top(y - x) \le 0 \, \forall y \in B_\infty$. Hence, $N(x)$ is a cone.
Moreover, for any $u, v \in N(x)$ and $\alpha \in (0, 1)$, we have $(\alpha u + (1-\alpha)v)^\top(y - x) \le 0$ for all $y \in B_\infty$. It follows that $N(x)$ is convex.
(b) $I_0 = \{i : -1 < x_i < 1\}$, $I_+ = \{i : x_i = 1\}$, $I_- = \{i : x_i = -1\}$.
Define $S(x) = \left\{u \in \mathbb{R}^n : \begin{cases} u_i = 0 & \text{if } i \in I_0, \\ u_i \ge 0 & \text{if } i \in I_+, \\ u_i \le 0 & \text{if } i \in I_- \end{cases}\right\}$. We claim $N(x) = S(x)$:
Suppose $u \in S(x)$. Then, for each $y \in B_\infty$, $|y_i| \le 1$ for $i = 1, \ldots, n$, implies $u^\top(y - x) = \sum_{i \in I_0} u_i(y_i - x_i) + \sum_{i \in I_+} u_i(y_i - 1) + \sum_{i \in I_-} u_i(y_i + 1) \le 0$. It follows that $u \in N(x)$.
Conversely, suppose $u \in N(x)$. Let $i \in \{1, \ldots, n\}$ fixed and $\alpha \in [-1, 1]$ arbitrary. Define $[y(i, \alpha)]_j = \begin{cases} x_j & \text{if } j \ne i, \\ \alpha & \text{otherwise.} \end{cases}$
Since $x \in B_\infty$, $y(i, \alpha) \in B_\infty$. This together with definition of u yields $u^\top(y(i, \alpha) - x) = u_i(\alpha - x_i) \le 0$. Since the preceding inequality holds for any $\alpha \in [-1, 1]$, we must have $u_i = 0$ if $i \in I_0$, $u_i \ge 0$ if $i \in I_+$, and $u_i \le 0$ if $i \in I_-$. It follows that $u \in S(x)$. This completes the proof.
2. Give explicit expression of f^*, the conjugate of f.
(a) Let $f(x) = \begin{cases} x \ln x & \text{if } x \ge 0, \\ +\infty & \text{otherwise} \end{cases}$ (note that $0 \ln 0 = 0$). (b) $f(x) = |x|$.
(c) Let $C = \{x \in \mathbb{R}^n_+ : \|x\|_2 \le 1\}$ and i_C the indicator function of C.
(d) Let $C \subseteq \mathbb{R}^n$ be a convex cone and i_C be the indicator function of C.
(e) $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x) = c^\top x + d$.
(f) $f(x) = \sum_{i=1}^n - \ln x_i$. (g) $f(x) = \sum_{i=1}^n |x_i|$.
A: (a) $f^*(y) = \sup_{x \in \mathbb{R}}\{yx - f(x)\} = \sup_{x \ge 0}\{yx - x \ln x\}$.
$x \mapsto yx - x \ln x$ is maximized at $x^* = \exp(y - 1) > 0$.
$f^*(y) = y \exp(y - 1) - (y - 1)\exp(y - 1) = \exp(y - 1)$.
(b) $f^*(y) = \sup_{x \in \mathbb{R}}\{yx - |x|\} = \sup_{x \in \mathbb{R}}\{|y||x| - |x|\}$.
$f^*(y) = \begin{cases} 0 & \text{if } |y| \le 1, \\ +\infty & \text{otherwise.} \end{cases}$
(c) $i_C^*(y) = \sup_{x \in \mathbb{R}^n}\{y^\top x - i_C(x)\} = \sup_{x \in C} y^\top x$.
Now, let $x \in C$ and $y \in \mathbb{R}^n$ be arbitrary.
$y^\top x = \sum_{i: y_i \ge 0} x_i y_i + \sum_{i: y_i < 0} x_i y_i \le \sum_{i: y_i \ge 0} x_i y_i = y^\top x' = y_+^\top x$,
where $x'_j = \begin{cases} x_j & \text{if } y_j \ge 0, \\ 0 & \text{otherwise,} \end{cases}$ and $(y_+)_j = \max\{y_j, 0\}$, for $j = 1, \ldots, n$.
It follows from the Cauchy-Schwarz inequality that $i_C^*(y) = \|y_+\|_2$.
(d) $i_C^*(y) = \sup_{x \in \mathbb{R}^n}\{y^\top x + i_C(x)\} = \sup_{x \in C} y^\top x$. Since C is a cone, we have $\alpha x \in C$ for all $\alpha > 0$ whenever $x \in C$. It follows that $i_C^*(y) = \begin{cases} 0 & \text{if } y^\top x \le 0 \text{ for all } x \in C, \\ +\infty & \text{otherwise.} \end{cases}$ (negative dual cone)
(e) By definition, we have $f^*(y) = \sup_{x \in \mathbb{R}^n}(y - c)^\top x - d$. Observe $\sup_{x \in \mathbb{R}^n}(y - c)^\top x = \begin{cases} 0 & \text{if } y = c, \\ +\infty & \text{otherwise.} \end{cases}$ It follows $f^*(y) = \begin{cases} -d & \text{if } y = c, \\ +\infty & \text{otherwise.} \end{cases}$
(f) $f^*(y) = \sup_{x \in \mathbb{R}^n_+}\left\{y^\top x + \sum_{i=1}^n \ln x_i\right\} = \sup_{x \in \mathbb{R}^n_+} \sum_{i=1}^n (y_i x_i + \ln x_i)$.
$\sup_{x_i \in \mathbb{R}_+}(y_i x_i + \ln x_i) = \begin{cases} -1 - \ln(-y_i) & \text{if } y_i < 0, \\ +\infty & \text{otherwise.} \end{cases}$
$f^*(y) = \begin{cases} -\sum_{i=1}^n \ln(-y_i) - n & \text{if } y \in \mathbb{R}^n_{--}, \\ +\infty & \text{otherwise.} \end{cases}$
(g) $f^*(y) = i_B y$, where $B = \{y \in \mathbb{R}^n : |y_i| \le 1 \text{ for } i = 1 \ldots n\}$
3. Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ be given. Consider the polyhedron $P = \{x \in \mathbb{R}^n : Ax \le b\}$. For any $x \in P$, define $N(x) = \{u \in \mathbb{R}^n : u^\top(y - x) \le 0 \text{ for all } y \in P\}$, $Q(x) = \{A^\top z \in \mathbb{R}^n : z^\top(b - Ax) = 0, z \in \mathbb{R}^m_+\}$.
(a) Show that $Q(x) \subseteq N(x)$.
A: Let $u \in Q(x)$. Then, we have $u = A^\top z$ for some $z \in \mathbb{R}^m$ satisfying $z^\top(b - Ax) = 0$ and $z \ge 0$. Now, for each $y \in P$, we get $z^\top A(y - x) \le z^\top(b - Ax) = 0$, (inequality from $z \ge 0$ and $Ay \le b$).
(b) Let $u \in \mathbb{R}^n$ s.t. $u \notin Q(x)$. Let $I(x) = \{i : a_i^\top x = b_i\}$, where $a_i^\top$ is the i-th row of A. Show $\exists w \in \mathbb{R}^n \setminus \{0\}$ satisfying $w^\top a_i \le 0 \, \forall i \in I(x)$.
A: If $I(x) = \emptyset$, then every $w \in \mathbb{R}^n \setminus \{0\}$ satisfies the desired conclusion. Assume $I(x) \ne \emptyset$. Since $Q(x)$ is a non-empty closed convex set, by the separation theorem and the definition of $Q(x)$, $\exists w \in \mathbb{R}^n \setminus \{0\}$ s.t.
$\theta^* \triangleq \max\left\{w^\top A^\top z : z^\top(b - Ax) = 0, z \in \mathbb{R}^m_+\right\} < w^\top u$. (3).
Since $\{z \in \mathbb{R}^m : z^\top(b - Ax) = 0, z \ge 0\}$ is a cone containing the origin and RHS of (3) is finite, $\theta^* = 0$. Now, for each $i \in I(x)$, the i-th basis vector $e_i \in \mathbb{R}^m$ satisfies $e_i^\top(b - Ax) = 0$ and $e_i \ge 0$. It follows $w^\top A^\top e_i = w^\top a_i \le \theta^*$ for all $i \in I(x)$.
(c) Show that for some $\epsilon > 0$, we have $x + \epsilon w \in P$, where w is the vector found in (b). Hence, conclude that $u \notin N(x)$.
A: *Case 1:* $i \in I(x)$. (b) implies $a_i^\top(x + \epsilon w) \le b_i$ for all $\epsilon > 0$.
Case 2: $i \notin I(x)$. Since $x \in P$, we have $a_i^\top x < b_i$. If $w^\top a_i \le 0$, we have $a_i^\top(x + \epsilon w) < b_i$ for all $\epsilon > 0$. On the other hand, upon letting $\bar{\epsilon} \triangleq \ldots > 0$, we see that if $w^\top a_i > 0$, then $a_i^\top(x + \epsilon w) \le b_i$ for all $\epsilon \in (0, \bar{\epsilon}]$.
Putting the above two cases together, we conclude $x + \bar{\epsilon} w \in P$. Finally, using (3), we have $u^\top((x + \bar{\epsilon} w) - x) = \bar{\epsilon} u^\top w > 0$. It follows that $u \notin N(x)$.
4. Let $K \subseteq \mathbb{R}^n$ be a closed convex cone and $x \in \mathbb{R}^n$ satisfying $x \notin K$. Show there exists $b \in \mathbb{R}^n$ satisfying $b^\top w \le 0 < b^\top x$ for all $w \in K$.
A: By separation theorem, $\exists b : \max_{w \in K} b^\top w < b^\top x$ $(*)$
Note that $0 \in K$. $\Rightarrow \max_{w \in K} b^\top w \ge 0$. We claim $\max_{w \in K} b^\top w = 0$.
Suppose not. Then, $\exists \bar{w} \in K$ s.t. $b^\top \bar{w} > 0$.
Since $\alpha \bar{w} \in K \, \forall \alpha > 0$, then $\alpha b^\top \bar{w} < b^\top x \, \forall \alpha > 0$.
Due to $*$. Taking $\alpha \to +\infty$ leads to a contradiction.
5. **23Q4.** $\mathbb{S}_d = \{X \in \mathbb{R}^{d \times d} : X^\top = X\}$. Let $\lambda_1(X) \ge \lambda_2(X) \ge \ldots \ge \lambda_d(X)$ denote the eigenvalues of $X \in \mathbb{S}_d$ in descending order. (a) Let $k \le d$, show $\sum_{i=1}^k \lambda_i(X) = \sup_{V \in \mathbb{R}^{d \times k}} \mathrm{Tr}(V^\top X V)$ s.t. $V^\top V = I_k$.
(b) Show the function is convex over $X \in \mathbb{S}^d$: $f(X) = \sum_{i=1}^k \lambda_i(X)$.
A: (a) Since X is symmetric, exists an orthogonal matrix $U \in \mathbb{R}^{d \times d}$ such that $X = U \Lambda U^\top$, where $\Lambda = diag(\lambda_1, \ldots, \lambda_d)$ Take V as the first k columns of U, i.e., $V = U_{:,1:k}$. Then, we have $\mathrm{tr}[V^\top X V] = \sum_{i=1}^k \lambda_i(X)$, which implies that $\sum_{i=1}^k \lambda_i(X) \le \sup_{V \in \mathbb{R}^{d \times k}} \mathrm{Tr}(V^\top X V)$ s.t. $V^\top V = I_k$. Also we have $\mathrm{Tr}(V^\top X V) = \mathrm{Tr}(V^\top U \Lambda U^\top V)$. Let $W = U^\top V \in \mathbb{R}^{d \times k}$, $\mathrm{Tr}(V^\top X V) = \mathrm{Tr}(W^\top \Lambda W)$. Let $W = [w_1^\top; \ldots; w_d^\top]$ Since Λ is diagonal, $\mathrm{Tr}(W^\top \Lambda W) = \sum_{i=1}^d \lambda_i \|w_i\|^2$. Since $W^\top W = I_k$, we have $\sum_{i=1}^d \|w_i\|^2 \le k$ and $\|w_i\|^2 \le 1$. Thus, $\sup \sum_{i=1}^d \lambda_i \|w_i\|^2 \le \sup \sum_{i=1}^d \lambda_i$.
(b) From (a) we know $f(X)$ is the pointwise supremum of a family of linear functions $\mathrm{tr}(V^\top X V)$.
6. **22Q1.** $f : \mathbb{R}^d \to \mathbb{R}$ convex and differentiable. Suppose exists $L > 0$ s.t. $0 \le f(x) - f(y) - \nabla f(y)^\top(x - y) \le \frac{L}{2}\|x - y\|^2$, $\forall x, y$. Show: $f(x) - f(y) \le \nabla f(x)^\top(x - y) - \frac{1}{2L}\|\nabla f(x) - \nabla f(y)\|^2$, $\forall x, y$.
A: Set $z = y - \frac{1}{L}(\nabla f(y) - \nabla f(x)) \Rightarrow \nabla f(y) - \nabla f(x) = L(y - z)(*)$.
$f(x) - f(y) \le \nabla f(x)^\top(x - y) - \frac{1}{2L}\|y - z\|^2$
$\Leftrightarrow f(x) - f(z) + f(z) - f(y) \le \nabla f(x)^\top(x - y) - \frac{L}{2}\|y - z\|^2$
$\Leftrightarrow f(x) - f(z) + f(z) - f(y) \le \nabla f(x)^\top(x - z) + \nabla f(x)^\top(z - y) - \frac{L}{2}\|y - z\|^2$
From convexity: $f(x) - f(z) \le \nabla f(x)^\top(x - z)$. (1)
From statement: $f(z) - f(y) - \nabla f(y)^\top(z - y) \le \frac{L}{2}\|z - y\|^2$ $\overset{(*)\nabla f(y) = \nabla f(x) + L(y - z)}{\Longrightarrow}$
$f(z) - f(y) \le \nabla f(x)^\top(z - y) - \frac{L}{2}\|z - y\|^2$ (2)
$(1) + (2)$: $f(x) - f(y) \le \nabla f(x)^\top(x - y) - \frac{L}{2}\|y - z\|^2 \overset{(*)}{=}$ □
7. **22Q3.** Consider a function $f : \mathbb{R}^d \to \mathbb{R}$. The function f is said to be quasi-convex if all of its sublevel sets are convex sets, i.e., $S_\alpha = \{x \in \mathbb{R}^d : f(x) \le \alpha\}$ is a convex set for any $\alpha \in \mathbb{R}$.
(a) Consider the function $f(x) = \frac{a^\top x + b}{c^\top x + d}$, where a, b, c, d are some fixed vectors/scalars. Let $X = \{x \in \mathbb{R}^d : c^\top x + d \ge 1\}$. Is $f(x)$ convex over X? Is $f(x)$ quasi-convex over X? Justify your answer.
(b) Show that for any quasi-convex function, it holds $f(\alpha x + (1 - \alpha)y) \le \max\{f(x), f(y)\}$, for any $x, y \in \mathbb{R}^d$ and $\alpha \in [0, 1]$.
A: (a) $S_\alpha = \left\{x \in \mathbb{R}^d : \frac{a^\top x + b}{c^\top x + d} \le \alpha\right\} = \{x \in \mathbb{R}^d : a^\top x + b \le \alpha(c^\top x + d)\} = \{x \in \mathbb{R}^d : (a - \alpha c)^\top x \le \alpha d - b\}$, which represents a halfspace (a convex set). Therefore, $f(x)$ is quasi-convex over X.
(b) Assume there exists y, z and $\alpha \in [0, 1]$ such that $f(\alpha y + (1 - \alpha)z) > \max\{f(y), f(z)\}$. Consider sublevel set $S_{\max\{f(y), f(z)\}} = \{x \in \mathbb{R}^d : f(x) \le \max\{f(y), f(z)\}\}$. Then $y, z \in S_{\max\{f(y), f(z)\}}$, but $\alpha y + (1 - \alpha)z \notin S_{\max\{f(y), f(z)\}}$, contradicts quasi-convexity of f.
8. **Geometric mean of k smallest eigenvalues.**
$f(X) = \left(\prod_{i=n-k+1}^n \lambda_i(X)\right)^{1/k}$ is concave on S^n_{++}: For $X \succ 0$, $f(X) = \frac{1}{k}\inf\{\mathrm{tr}(V^\top X V) \mid V \in \mathbb{R}^{n \times k}, \det V^\top V = 1\}$. f is the pointwise infimum of a family of linear functions $\mathrm{tr}(V^\top X V)$.
9. **Log of product of k smallest eigenvalues.**
$\sum_{i=n-k+1}^n \log \lambda_i(X)$ is concave on S^n_{++}: For $X \succ 0$, $\prod_{i=n-k+1}^n \lambda_i(X) = \inf\left\{\prod_{i=1}^k (V^\top X V)_{ii} \mid V \in \mathbb{R}^{n \times k}, V^\top V = I\right\}$
f is pointwise infimum of a family of concave functions $\log \prod_i (V^\top X V)_{ii} = \sum_i \log(V^\top X V)_{ii}$.
10. $\lambda_1^k : S^n \to \mathbb{R}$ returns the sum of the k largest eigenvalues of its argument.
(a) Show $\lambda_1^k(A) = \max\ \mathrm{tr}(AX)$ s.t. $\mathrm{tr}(X) = k$ $I \succeq X \succeq 0$.
(b) Show λ_1^k is convex for each $k \ge 1$.
A: (a) $\mathrm{tr}(AX) = \mathrm{tr}(U \Lambda U^\top X) = \mathrm{tr}(\Lambda U^\top X U)$. Since $\mathrm{tr}(X) = \mathrm{tr}(XUU^\top) = \mathrm{tr}(U^\top X U)$ and $v^\top X v = (U^\top v)^\top (U^\top X U)(U^\top v)$ for any $v \in \mathbb{R}^n$, we see that $X \in U_k$ iff $U^\top X U \in U_k$, where $U_k \equiv \{Z \in S^n : \mathrm{tr}(Z) = k, I \succeq Z \succeq 0\}$.
The given problem is equivalent to $\mathrm{tr}(\Lambda X) = \mathrm{tr}(XUU^\top) = \mathrm{tr}(U^\top X U)$.
Now, we claim that there exists an optimal solution to (1) that is diagonal. To see this, observe $\mathrm{tr}(\Lambda X) = \sum_{i=1}^n \Lambda_{ii} X_{ii}$, and $I \succeq X \succeq 0$ implies that $X_{ii} \in [0, 1]$ for $i = 1, 2, \ldots, n$. In particular, if X^* is optimal, then diagonal matrix $\bar{X}^* = \mathrm{diag}(X_{11}^*, X_{22}^*, \ldots, X_{nn}^*)$ is feasible and has the same objective value as X^*. This establishes the claim. Consequently, equivalent to $\max \sum_{i=1}^n \Lambda_{ii} x_i$ s.t. $\sum_{i=1}^n x_i = k$, $0 \le x \le e$.
(b) $f_X : S^n \to \mathbb{R}$ by $f_X(A) = \mathrm{tr}(AX)$. $\lambda_1^k(A) = \max_{X \in U_k} f_X(A)$; i.e., λ_1^k is the pointwise supremum of a collection of linear functions.
11. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function s.t. $\mathrm{epi}(f)$ is closed and f is not identically $+\infty$. (a) Show $f = f^{**}$, where $f^{**} = (f^*)^*$ is the conjugate of f^*. (b) Show for any $x, y \in \mathbb{R}^n$, the following statements are equivalent: (i) $y \in \partial f(x)$; (ii) $f(x) + f^*(y) = x^\top y$; (iii) $x \in \partial f^*(y)$.
A: (a) $f(x) = \sup_{(y,c) \in S_f}\{y^\top x - c\}$, where $S_f = \{(y, c) \in \mathbb{R}^n \times \mathbb{R} : y^\top x - c \le f(x) \text{ for all } x \in \mathbb{R}^n\}$. Moreover, $S_f = \mathrm{epi}(f^*)$. Hence, we have $(y, c) \in S_f$ iff $f^*(y) \le c$, implies $f(x) = \sup_{y \in \mathbb{R}^n}\{y^\top x - f^*(y)\}$,.
(b) Suppose that (i) holds; i.e., $y \in \partial f(x)$, $f(z) \ge f(x) + y^\top(z - x)$ for all $z \in \mathbb{R}^n$, $y^\top x - f(x) \ge y^\top z - f(z)$ for all $z \in \mathbb{R}^n$. In particular, we have $y^\top x - f(x) \ge \sup_{z \in \mathbb{R}^n}\{y^\top z - f(z)\} = f^*(y)$. On the other hand, $f(x) \ge y^\top x - f^*(y)$. Hence, $f(x) + f^*(y) = y^\top x$; i.e., (ii) holds. By reversing the argument, the converse also holds.
Next, suppose (ii) holds; i.e., $f(x) + f^*(y) = y^\top x$. By result in (a), $f^{**}(x) + f^*(y) = x^\top y$. Since $f^{**}(x) \ge z^\top x - f^*(z)$ for all $z \in \mathbb{R}^n$, we obtain $y^\top x \ge f^*(y) + z^\top x - f^*(z)$ for all $z \in \mathbb{R}^n$, or equivalently, $f^*(z) \ge f^*(y) + x^\top(z - y)$ for all $z \in \mathbb{R}^n$. This shows that $x \in \partial f^*(y)$; i.e., (iii) holds. Again, the converse follows by reversing.
12. $S = \{X \in S^n : \lambda_{\max}(X) \le 1, \ X \succeq 0\} = \{X \in S^n : I \succeq X \succeq 0\}$ convex.
$S = \{X \in S^n : \mathrm{rank}(X) \le 1\}$ not convex: Let $X_1 = e_1 e_1^\top$ and $X_2 = e_2 e_2^\top$.

4 Linear Programming

4.1 Basic Definitions and Properties

Definition 1 Let $s \in \mathbb{R}^n \setminus \{0\}$ and $c \in \mathbb{R}$ be given. Then, the set of solutions to the linear equation $s^\top x = c$, namely, $H = \{x \in \mathbb{R}^n : s^\top x = c\}$, is called a **hyperplane** in $\mathbb{R}^n$. Associated with every hyperplane H are the two **halfspaces** $H^- = \{x \in \mathbb{R}^n : s^\top x \le c\}$ and $H^+ = \{x \in \mathbb{R}^n : s^\top x \ge c\}$. s is a normal of H.
$H = H^+ \cap H^-$; $\mathbb{R}^n = H^+ \cup H^-$. H, H^+, H^- are all closed convex sets.
Geometrically, a hyperplane is an $(n-1)$-dimensional affine subspace; i.e., $H = \{\bar{x}\} + V$ where $V = \{x \in \mathbb{R}^n : s^\top x = 0\}$, $\bar{x} = \frac{c}{s^\top s} s$.
Proof: Since V is the set of vectors that are orthogonal to s, it is a linear subspace of dimension $n - 1$. Moreover, a simple calculation shows that $s^\top \bar{x} = c$ (i.e., $\bar{x} \in H$) and $\bar{x} + x \in H$ for any $x \in V$. Thus, $H \supseteq \{\bar{x}\} + V$. Conversely, for any $y \in H$, we have $x = y - \bar{x} \in V$, which implies that $H \subseteq \{\bar{x}\} + V$. It follows that $H = \{\bar{x}\} + V$, as desired. □
Definition 2 A **polyhedron** is the intersection of a finite set of halfspaces. A bounded polyhedron is called a **polytope**.
In particular, a closed convex set P is a polyhedron iff can be represented as $P = \{x \in \mathbb{R}^n : a_i^\top x \le b_i \text{ for } i = 1, \ldots, m\}$ (1)
for some given $a_1, \ldots, a_m \in \mathbb{R}^n$ and $b_1, \ldots, b_m \in \mathbb{R}$.

4.2 Extremal Elements of a Polyhedron

Consider now a point $\bar{x} \in P$, where $P \subseteq \mathbb{R}^n$ is a polyhedron of the form (1). If the index $i \in \{1, \ldots, m\}$ is such that $a_i^\top \bar{x} = b_i$, then we say that the corresponding constraint is active or binding at $\bar{x}$.
Theorem 1 Let $P \subseteq \mathbb{R}^n$ be a polyhedron of the form (1), and consider a point $\bar{x} \in P$. Let $I = \{i : a_i^\top \bar{x} = b_i\}$ be the set of indices of constraints that are active at $\bar{x}$. Then, the following are equivalent:
(a)There exist n vectors in set $\{a_i \in \mathbb{R}^n : i \in I\}$ that linearly independent.
(b)The point $\bar{x} \in \mathbb{R}^n$ is the unique solution to the following system of linear equations: $a_i^\top x = b_i$ for $i \in I$.
Definition 3 Let P be a polyhedron and $x \in \mathbb{R}^n$ be arbitrary. The vector x is called a **basic solution** if there are n linearly independent active constraints at x. If in addition we have $x \in P$, then we say that x is a **basic feasible solution**.
An **extreme point** is a point that does not lie strictly within a line segment connecting two other points of the set.
Theorem 2 Let $P \subseteq \mathbb{R}^n$ be a polyhedron of the form (1) and $x \in P$ be arbitrary. Then, the following are equivalent:
(a) x is an extreme point. (b) x is a basic feasible solution.
Remark: Also use **vertex** to mean an extreme point/basic feasible solution.
Proof Suppose $x \in P$ not a BFS. Let $I = \{i : a_i^\top x = b_i\}$. Then, the family $\{a_i \in \mathbb{R}^n : i \in I\}$ does not contain n linearly independent vectors. Hence, there exists a non-zero vector $d \in \mathbb{R}^n$ such that $a_i^\top d = 0$ for all $i \in I$. Now, let $\epsilon > 0$ be a parameter to be determined, and set $x_1 = x - \epsilon d \in \mathbb{R}^n$ and $x_2 = x + \epsilon d \in \mathbb{R}^n$. Clearly, for any $i \in I$, we have $a_i^\top x_1 = a_i^\top x_2 = a_i^\top x = b_i$. Moreover, for any $i \notin I$, we have $a_i^\top x < b_i$ because $x \in P$. It follows that for sufficiently small $\epsilon > 0$, we have $a_i^\top x_1 < b_i$ and $a_i^\top x_2 < b_i$ for any $i \notin I$. Hence, $x_1, x_2 \in P$. Since $x = (x_1 + x_2)/2$ and $x_1 \ne x_2$, we conclude that x is not an extreme point.
Conversely, suppose that $x \in P$ is not an extreme point. Let $x_1, x_2 \in P$ be such that $x_1 \ne x_2$ and $x = (x_1 + x_2)/2$, and let $I = \{i : a_i^\top x = b_i\}$. Since $x_1, x_2 \in P$, we have $a_i^\top x_1 \le b_i$ and $a_i^\top x_2 \le b_i$ for $i = 1, \ldots, m$, which yields $a_i^\top x_1 = a_i^\top x_2 = a_i^\top x = b_i$ for all $i \in I$. This implies that the system of linear equations $a_i^\top z = b_i$ for $i \in I$ has more than one solution in $z \in \mathbb{R}^n$. Hence, by Theorem 1, x is not a basic feasible solution. □
Example 1 (Non-Polyhedrality of the Euclidean Ball)
Proof: Consider the $B(0, 1) \subset \mathbb{R}^n$, which is a closed convex set. Suppose $B(0, 1)$ is a polyhedron. Then, it admits a representation of the form (1). Observe that the maximum number of basic feasible solutions in such a representation is $\binom{m}{n}$, which is finite. By Theorem 2, this is also the maximum number of extreme points of $B(0, 1)$. However, this contradicts the result that the number of extreme points of $B(0, 1)$ is infinite. Thus, we conclude that $B(0, 1)$ is non-polyhedral. It is worth noting that $B(0, 1)$ can be written as the infinite intersection of halfspaces $\bigcap_{d \in \mathbb{R}^n, \|d\|_2 = 1}\{x \in \mathbb{R}^n : d^\top x \le 1\}$.
Definition 4 A polyhedron $P \subseteq \mathbb{R}^n$ contains a **line** if there exists a point $x \in P$ and a vector $d \in \mathbb{R}^n \setminus \{0\}$ such that $x + \alpha d \in P$ for all $\alpha \in \mathbb{R}$.
Theorem 3 Let $P \subseteq \mathbb{R}^n$ be a non-empty polyhedron of the form (1). Then, the following are equivalent: (a) P has at least one vertex. (b)P does not contain a line. (c) There exist n linearly independent vectors in $\{a_i\}_{i=1}^m$.

4.3 Existence of Optimal Solutions to Linear Programs

Now, let $P \subseteq \mathbb{R}^n$ be a non-empty polyhedron of the form (1) and $h \in \mathbb{R}^n$ be a given vector. Consider the LP $\min_{x \in P} h^\top x$. $(*)$
Theorem 4 Consider the LP $(*)$. Suppose that P has at least one vertex. Then, either the optimal value is $-\infty$, or there exists a vertex that is optimal.
Proof Before we proceed, let us introduce a definition. We say that $x \in P$ has rank $k \ge 0$ if there are exactly k linearly independent active constraints at x. Now, suppose that the optimal value is finite. Consider an $x \in P$ of rank $k < n$. Our goal is to show that there exists some $y \in P$ of greater rank and satisfies $h^\top y \le h^\top x$. We can then repeat until reach an optimal vertex.
As usual, let $I = \{i : a_i^\top x = b_i\}$. Since there are only $k < n$ linearly independent vectors in the family $\{a_i \in \mathbb{R}^n : i \in I\}$, there exists a $d \in \mathbb{R}^n \setminus \{0\}$ such that $a_i^\top d = 0$ for all $i \in I$. W.l.o.g, assume $h^\top d \le 0$.
Case 1: $h^\top d < 0$. Consider the half-line $\mathcal{L}_+ = \{x + \alpha d : \alpha > 0\}$. Clearly, for any $y \in \mathcal{L}_+$ and $i \in I$, we have $a_i^\top y = b_i$. Now, if $\mathcal{L}_+ \subseteq P$, then the optimal value would be $-\infty$, which we have assumed not to be the case. Hence, there exist a scalar $\bar{\alpha} > 0$ and an index $j \notin I$ such that $a_j^\top(x + \bar{\alpha} d) = b_j$.
Let $y = x + \bar{\alpha} d$. Then, we have $h^\top y < h^\top x$. Moreover, following the argument in the proof of Theorem 3, we see that the family $\{a_i : i \in I\} \cup \{a_j\}$ is linearly independent, which implies that $y \in P$ has rank at least $k + 1$.
Case 2: $h^\top d = 0$. Consider the line $\mathcal{L} = \{x + \alpha d : \alpha \in \mathbb{R}\}$. Since P does not contain a line, there exists a scalar $\bar{\alpha} \ne 0$ and an index $j \notin I$ such that $a_j^\top(x + \bar{\alpha} d) = b_j$. Let $y = x + \bar{\alpha} d$. Then, we have $h^\top y = h^\top x$, and the rank of $y \in P$ is at least $k + 1$.
In either case, we obtain a $y \in P$ whose rank is greater than that of $x \in P$ and satisfies $h^\top y \le h^\top x$. By repeating the above process, we will end up with a $z \in P$ whose rank is n (i.e., z is a vertex of P) and satisfies $h^\top z \le h^\top x$. To complete the proof of the theorem, let $z_1, \ldots, z_r$ be the vertices of P and set $i^* = \arg\min_{1 \le i \le r} h^\top z_i$. Our argument above shows that for every $x \in P$, there exists an $i \in \{1, \ldots, r\}$ such that $h^\top z_i \le h^\top x$. It follows that $h^\top z_{i^*} \le h^\top x$ for all $x \in P$; i.e., $z_{i^*} \in P$ is an optimal vertex. □
Consider the polyhedron $P' = \{(x^+, x^-, s) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m : a_i^\top(x^+ - x^-) + s_i = b_i \text{ for } i = 1, \ldots, m;\ x^+, x^-, s \ge 0\}$. Note if $x \in P$, then setting $x_i^+ = \begin{cases} x_i & \text{if } x_i \ge 0, \\ 0 & \text{otherwise,} \end{cases}$ $x_i^- = \begin{cases} 0 & \text{if } x_i \ge 0, \\ -x_i & \text{otherwise,} \end{cases}$

$s_j = b_j - a_j^\top x$ for $j = 1, \ldots, m$, we see that $(x^+, x^-, s) \in P'$.

Conversely, if $(x^+, x^-, s) \in P'$, then by setting $x = x^+ - x^-$, we have $\min_{x \in P} h^\top x = \min_{(x^+,x^-,s)\in P'} h^\top(x^+ - x^-)$; i.e., minimizing $h^\top x$ over P is equivalent to minimizing $h^\top(x^+ - x^-)$ over P'. Furthermore, note that the polyhedron P' does not contain a line, and thus by Theorem 3, P' has at least one vertex.

Corollary 1 Consider the LP (*). Suppose that P is non-empty. Then, either the optimal value is $-\infty$, or there exists an optimal solution.

Eg $\inf_{x \ge 1} x^{-1}$ shows nonlinear optimization need not have such a property.

To simplify, let $y = (x^+, x^-, s) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{2n+m}$. Define

$$A = \begin{bmatrix} a_1^\top & -a_1^\top & 1 & 0 & \cdots & 0 \\ a_2^\top & -a_2^\top & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_m^\top & -a_m^\top & 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{m \times (2n+m)},$$

$b = (b_1, \ldots, b_m) \in \mathbb{R}^m, \quad c = (h, -h, \mathbf{0}) \in \mathbb{R}^{2n+m}$.

Then, the problem of minimizing $h^\top(x^+ - x^-)$ over P' can be written as

$$\text{minimize } c^\top y \quad \text{subject to } Ay = b,\ y \ge 0. \tag{3}$$

We shall call an LP problem of the form (3) a **standard form problem**.

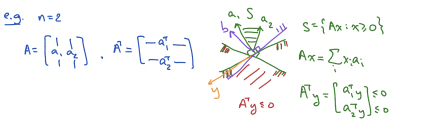
Example 2 (Conversion to Standard Form LP)

Let $P = \{x \in \mathbb{R}^2 : e_1^\top x \ge 1\} \subset \mathbb{R}^2$ and $h = e_1 \in \mathbb{R}^2$ in the LP (*). It is clear that $(1, x_2)$ is an optimal solution for any $x_2 \in \mathbb{R}$. The polyhedron P' is given by $P' = \{(x_1^+, x_2^+, x_1^-, x_2^-, s) \in \mathbb{R}^5_+ : x_1^+ - x_1^- - s = 1\} \subset \mathbb{R}^5$ and the LP (*) is equivalent to $\min_{(x_1^+,x_2^+,x_1^-,x_2^-,s)\in P'} x_1^+ - x_1^-$. Since P' has at least one vertex, by Theorem 4, the LP (4) has a vertex optimal solution. This is given by $y^* = (1, 0, 0, 0, 0)$. To verify y^* is a vertex of P', it suffices to verify the five active constraints $x_1^+ = 1, x_2^+ = 0, x_1^- = 0, x_2^- = 0, s = 0$ are linearly independent; see Theorem 2.

4.4 Theorems of Alternatives

Theorem 5 (Farkas' Lemma) Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ be given. Then, exactly one of the following systems has a solution:

$$Ax = b, \quad x \ge 0. \ (5) \qquad A^\top y \le 0, \quad b^\top y > 0. \ (6)$$



Corollary 2 (Gordan's Theorem) Let $A \in \mathbb{R}^{m\times n}$ be given. Then, exactly one of the following systems has a solution:

$$Ax > 0. \ (8) \qquad A^\top y = 0, \quad y \ge 0, \quad y \ne 0. \ (9)$$

Proof (S1) (8) and (9) cannot both have solutions, otherwise there would exist $\bar{x} \in \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}^m$ s.t. $0 = (\bar{y}^\top A)\bar{x} = \bar{y}^\top(Ax) > 0$, contradiction.

(S2) Now, note that (8) is equivalent to $Ax \ge e$, since we can scale both sides of (8) by any positive scalar. On the other hand, the system $Ax \ge e$ is equivalent to the system $\tilde{A}z = e,\ z \ge 0$, where $\tilde{A} = [A \ \ -A \ \ -I] \in \mathbb{R}^{m\times(2n+m)}$ and $z = (x^+, x^-, s) \in \mathbb{R}^{2n+m}_+$. Now, by Farkas' lemma, if the system $\tilde{A}z = e,\ z \ge 0$ has no solution, then there exists a $y \in \mathbb{R}^m$ such that $\tilde{A}^\top y \le 0$ and $e^\top y > 0$. From definition of $\tilde{A}$, we see $A^\top y = 0$ and $y \ge 0$. Moreover, since $e^\top y > 0$, we conclude $y \ne 0$. This completes the proof. □

Lemma in 21Q4: (I) $Ax > 0,\ x \ge 0$. (II) $y^\top A \le 0,\ y \ge 0,\ y \ne 0$.

Lemma in 23Q5: (I) $Ux = v,\ x \ge 0$. (II) $y^\top U \ge 0,\ y^\top v < 0$.

A list of equivalence tricks:

1. (Cor 2) $Ax > 0 \iff Ax \ge e \iff [A \ -A \ -I]z = e,\ z \ge 0$;
2. (Cor 2) $e^\top y > 0 \iff y \ne 0$;
3. (Thm 7, HW 1b) ">" or "<": Homogenize by switching x to $x/t,\ t \ge 0$;
4. (Thm 7, HW 1b, 17Q1) $Ax = bt \Leftrightarrow Ax - bt \ge 0, -Ax + bt \ge 0$ to match (II); or $Ax \le b \Leftrightarrow (A, -I)(x, s)^\top = b, (x, s) \ge 0$ to match (I).
5. (HW 1a) $Ax \le 0, Ax \ne 0 \iff Ax \le 0, e^\top Ax = -1$.

4.5 LP Duality Theory

$$v_p^* = \min c^\top x \quad \text{subject to } Ax = b,\ x \ge 0. \tag{P}$$

Suppose that we can find a vector $y \in \mathbb{R}^m$ such that $A^\top y \le c$. Then, for any $x \in \mathbb{R}^n$ that is feasible for (P), we have $b^\top y = x^\top A^\top y \le c^\top x$, where the equality is due to $Ax = b$ and the inequality is due to $x \ge 0$ and $A^\top y \le c$. Since the above inequality holds for any feasible solution $x \in \mathbb{R}^n$ to (P), it follows that $b^\top y$ provides a lower bound on v_p^* for any $y \in \mathbb{R}^m$ satisfying $A^\top y \le c$. Naturally, we are interested in finding the largest lower bound on v_p^*. This motivates us to consider the following optimization problem:

$$v_d^* = \max b^\top y \quad \text{subject to } A^\top y \le c. \tag{D}$$

Note that (D) is also an LP. In the sequel we shall call (P) the **primal problem** and (D) its **dual problem**. Our discussion above leads to the following result:

Theorem 6 (LP Weak Duality) Let $\bar{x} \in \mathbb{R}^n$ be feasible for (P) and $\bar{y} \in \mathbb{R}^m$ be feasible for (D). Then, we have $b^\top \bar{y} \le c^\top \bar{x}$. In particular, $v_p^* \ge v_d^*$.

Corollary 3 The following hold:

(a) If the optimal value of (P) is $-\infty$, then (D) must be infeasible.
(b) If the optimal value of (D) is $+\infty$, then (P) must be infeasible.
(c) Let $\bar{x} \in \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}^m$ be feasible for (P) and (D), respectively. Suppose that the duality gap $\Delta(\bar{x}, \bar{y}) = c^\top \bar{x} - b^\top \bar{y} = 0$. Then, $\bar{x}$ and $\bar{y}$ are optimal solutions to (P) and (D), respectively.

Note: It's possible for both (P) and (D) to be infeasible.

Theorem 7 (LP Strong Duality) Suppose (P) has an optimal solution $x^* \in \mathbb{R}^n$. Then, (D) also has an optimal solution $y^* \in \mathbb{R}^m, c^\top x^* = b^\top y^*$.

Proof: (S1) Suppose (P) has an optimal solution $x^* \in \mathbb{R}^n$. Then, the system

$$Ax = b, \quad x \ge 0, \quad c^\top x < c^\top x^* \tag{10}$$

does not have a solution in $x \in \mathbb{R}^n$.

(SH) To apply Farkas' lemma, we first homogenize the above system to get

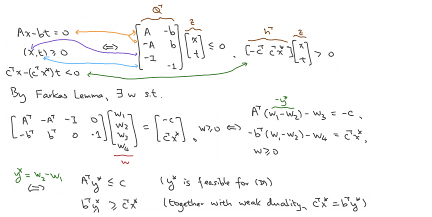
$$Ax - bt = 0, \quad c^\top x - (c^\top x^*)t = -1 < 0, \quad (x, t) \ge 0. \tag{11}$$

We claim that (11) has no solution in $(x, t) \in \mathbb{R}^n \times \mathbb{R}$. Indeed,

(i) if (x', t') is a feasible solution to (11) with $t' > 0$, then x'/t' is a solution to (10), which is a contradiction.

(ii) if $t' = 0$, then we have $A(x^* + x') = b,\ x^* + x' \ge 0,\ c^\top(x^* + x') = c^\top x^* - 1 < c^\top x^*$. This shows $x^* + x'$ is a solution to (10), which again is a contradiction. Thus, the claim is established.

(S2) By Farkas' lemma: $Qw = h, \quad w \ge 0. \qquad Q^\top z \le 0, \quad h^\top z > 0.$



Corollary 4 Suppose that both (P) and (D) are feasible. Then, both (P) and (D) have optimal solutions, and their respective optimal values are equal.

The task of finding optimal solutions to (P) and (D) is equivalent to finding a feasible solution to the following linear system in $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$:

$Ax = b, \quad x \ge 0, \quad$ (primal feasibility)
$A^\top y \le c, \quad$ (dual feasibility)
$c^\top x = b^\top y. \quad$ (zero duality gap),

the problem of *linear optimization* is no harder than that of *linear feasibility*.

Theorem 8 (Complementary Slackness) Let $\bar{x} \in \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}^m$ be feasible for (P) and (D), respectively. Then, the vectors $\bar{x}$ and $\bar{y}$ are optimal for their respective problems iff $\bar{x}_i(c - A^\top \bar{y})_i = 0$ for $i = 1, \ldots, n$.

Proof Using the fact that $A\bar{x} = b$, we have (12)
$c^\top \bar{x} - b^\top \bar{y} = c^\top \bar{x} - \bar{x}^\top A^\top \bar{y} = \bar{x}^\top(c - A^\top \bar{y}) = \sum_{i=1}^n \bar{x}_i(c - A^\top \bar{y})_i$.

Now, if $\bar{x}_i(c - A^\top \bar{y})_i = 0$ for $i = 1, \ldots, n$, then we have $c^\top \bar{x} = b^\top \bar{y}$. By the LP strong duality theorem, we conclude that $\bar{x}$ and $\bar{y}$ are optimal for their respective problems. Conversely, if $\bar{x}$ and $\bar{y}$ are optimal for their respective problems, then by the LP strong duality theorem, we have $c^\top \bar{x} - b^\top \bar{y} = 0$. Since $\bar{x} \ge 0$ and $c - A^\top \bar{y} \ge 0$ by the feasibility of $\bar{x}$ and $\bar{y}$, we conclude by (12) that $\bar{x}_i(c - A^\top \bar{y})_i = 0$ for $i = 1, \ldots, n$. □

From Theorem 8, we see that another way of solving (P) and (D) is to solve the following (nonlinear) system in $(x, y, s) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$:

$Ax = b, \quad x \ge 0, \quad$ (primal feasibility)
$A^\top y + s = c, \quad$ (dual feasibility)
$x_i s_i = 0$ for $i = 1, \ldots, n. \quad$ (complementarity)

Example 3 (A Simple LP) Consider the following LP:

$$\begin{aligned} \text{minimize} \quad & x_1 + 2x_2 + x_3 \\ \text{subject to} \quad & x_1 - 2x_2 + x_3 \ge 2, \\ & -x_1 + x_3 \ge 4, \\ & 2x_1 + x_3 \ge 6, \\ & x_1 + x_2 + x_3 \ge 2, \\ & x \ge 0. \end{aligned}$$

To derive the dual, we put it into standard form:

$$\begin{aligned} \text{minimize} \quad & (1, 2, 1, 0, 0, 0, 0)^\top (x_1, x_2, x_3, s_1, s_2, s_3, s_4) \\ \text{subject to} \quad & \begin{bmatrix} 1 & -2 & 1 & \\ -1 & 0 & 1 & -I \\ 2 & 0 & 1 & \\ 1 & 1 & 1 & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ s \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 2 \end{bmatrix} \\ & (x, s) \ge 0. \end{aligned}$$

Dual:

$$\begin{aligned} \text{maximize} \quad & (2, 4, 6, 2)^\top (y_1, y_2, y_3, y_4) \\ \text{subject to} \quad & \begin{bmatrix} 1 & -1 & 2 & 1 \\ -2 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ & -I & & \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \le \begin{bmatrix} 1 \\ 2 \\ 1 \\ \mathbf{0} \end{bmatrix}. \end{aligned}$$

Now, point $(\bar{x}, \bar{s}) = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{s}_1, \bar{s}_2, \bar{s}_3, \bar{s}_4) = \left(\frac{2}{3}, 0, \frac{14}{3}, 0, 0, \frac{10}{3}\right)$, feasible for Primal. By Theorem 8, the point $(\bar{x}, \bar{s})$ is optimal for Primal iff there exists a feasible solution $\bar{y} \in \mathbb{R}^4$ to Dual such that

$\bar{y}_1 = \bar{y}_4 = 0, \quad$ (since $\bar{s}_1, \bar{s}_4 > 0$) $\quad -\bar{y}_2 + 2\bar{y}_3 = 1 \quad$ (since $\bar{x}_1 > 0$)
$\bar{y}_2 + \bar{y}_3 = 1, \quad$ (since $\bar{x}_3 > 0$) $\quad \bar{y}_2, \bar{y}_3 \ge 0, \quad$ (dual feasibility)

or equivalently, point $\bar{y} = \left(0, \frac{1}{3}, \frac{2}{3}, 0\right)$ is feasible for Dual (easily verified).

Hence, we certified the optimality of primal-dual pair of solutions $(\bar{x}, \bar{s}, \bar{y})$. Note we also have $(1, 2, 1, 0, 0, 0, 0)^\top(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{s}_1, \bar{s}_2, \bar{s}_3, \bar{s}_4)$ $= \frac{16}{3} = (2, 4, 6, 2)^\top(\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4)$; i.e., the duality gap is zero.

By Theorems 4 and 6, we know that Problem (13) has a vertex optimal solution. Recall from Definition 3 that each vertex of the feasible region of Problem should have three linearly independent active constraints. $(-1, 0, 1)^\top(\bar{x}_1, \bar{x}_2, \bar{x}_3) = 4, \quad (2, 0, 1)^\top(\bar{x}_1, \bar{x}_2, \bar{x}_3) = 6,$ $(0, 1, 0)^\top(\bar{x}_1, \bar{x}_2, \bar{x}_3) = 0$, are linearly independent (they correspond to the coefficient vectors $(-1, 0, 1), (2, 0, 1), (0, 1, 0)$), we conclude that $\bar{x}$ is a vertex optimal solution to Problem.

4.6 Conclusion on Optimality Conditions

$$\begin{array}{llll} \min & c^\top x & & \max & b^\top y & \\ \text{s.t.} & Ax = b, & (P) & \text{s.t.} & A^\top y + s = c, & (D) \\ & x \ge \mathbf{0}, & & & s \ge \mathbf{0}, & \end{array}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$ are given. The solutions x^* and (y^*, s^*) are optimal for (P) and (D), respectively, iff they satisfy the following optimality conditions:

$x_i^* s_i^* = 0$	for $i = 1, \ldots, n$,	(complementarity)
$Ax^* = b, \quad x^* \ge 0,$		(primal feasibility)
$A^\top y^* + s^* = c, \quad s^* \ge 0.$		(dual feasibility)

4.7 An Approximation Algorithm for Vertex Cover

Consider a simple undirected graph $G = (V, E)$, where each vertex $v_i \in V$ has an associated cost $c_i \in \mathbb{R}_+$. A <u>vertex cover</u> of G is a subset $S \subset V$ such that for every edge $(v_i, v_j) \in E$, at least one of the endpoints belongs to S. We are interested in finding a vertex cover S of G of minimal cost.

Now, let $x_i \in \{0, 1\}$ be a binary variable indicating whether v_i belongs to the vertex cover S or not (i.e., $x_i = 1$ iff $v_i \in S$). Then, the minimum–cost vertex cover problem can be formulated as the following integer program:

$$v^* = \min c^\top x = \sum_{i \in V} c_i x_i \quad \text{s.t.} \quad x_i + x_j \ge 1 \quad \text{for} \quad (v_i, v_j) \in E,$$
$$x \in \{0, 1\}^{|V|}.$$

Using the fact that $c \ge 0$, it is not hard to show that the resulting problem is equivalent to the following LP, which is called an *LP relaxation* of Problem:

$$v_r^* = \min \quad c^\top x \quad \text{s.t.} \quad x_i + x_j \ge 1 \quad \text{for} \quad (v_i, v_j) \in E,$$
$$x \ge 0.$$

Clearly, we have $v_r^* \le v^*$. Suppose that x' is an optimal solution to Problem (7). It is then natural to ask whether we can convert x' into a solution x'' that is feasible for Problem (6) and satisfies $c^\top x'' \le \alpha v^*$ for some $\alpha > 0$. The key to proving this is the following theorem:

Theorem 3 Let $P \subseteq \mathbb{R}^{|V|}$ be the polyhedron defined by the following system: $\begin{cases} x_i + x_j & \ge 1 \quad \text{for } (v_i, v_j) \in E, \\ x & \ge 0. \end{cases}$ Suppose that x is an extreme point of P. Then, we have $x_i \in \{0, 1/2, 1\}$ for $i = 1, \ldots, |V|$.

Proof Let $x \in P$ and consider the sets
$U_{-1} = \{i \in \{1, \ldots, |V|\} : x_i \in (0, 1/2)\}$;
$U_1 = \{i \in \{1, \ldots, |V|\} : x_i \in (1/2, 1)\}$.
For $i = 1, \ldots, |V|$ and $k \in \{-1, 1\}$, define

$$y_i = \begin{cases} x_i + k\epsilon & \text{if } i \in U_k, \\ x_i & \text{otherwise} \end{cases}, \quad z_i = \begin{cases} x_i - k\epsilon & \text{if } i \in U_k, \\ x_i & \text{otherwise.} \end{cases}$$

By definition, we have $x = (y + z)/2$. If either U_{-1} or U_1 is non–empty, then we may choose $\epsilon \downarrow 0$ so that $y, z \in P$, and that x, y, z are all distinct. It follows that $U_k = \emptyset$ for $k \in \{-1, 1\}$ if x is an extreme point of P. □

Corollary 1 There exists a 2–approximation algorithm for the minimum–cost vertex cover problem.

Proof We first solve the LP and obtain an optimal extreme point solution x'. Now, by Theorem 3, all entries of x' belong to $\{0, 1/2, 1\}$. Hence, the vector x'' defined by $x_i'' = \begin{cases} x_i' & \text{if } x_i' = 0 \text{ or } 1, \\ 1 & \text{if } x_i' = 1/2 \end{cases}$ for $i = 1, \ldots, |V|$ is feasible for Problem. Moreover, objective value $c^\top x'' \le 2c^\top x' = 2v_r^* \le 2v^*$. □

4.8 Example Problems

1. **1. Farkas Lemma:**
(a) (I) $\quad Ax \le 0,\ Ax \ne 0,\ x \ge 0. \qquad$ (II) $\quad A^\top y \ge 0,\ y > 0.$
A: (S1) The systems (I) and (II) cannot be simultaneously solvable. Indeed, suppose that $\bar{x} \in \mathbb{R}^n$ solves (I) and $\bar{y} \in \mathbb{R}^m$ solves (II). Then, since $\bar{y} > 0$, $A\bar{x} \le 0$ and $A\bar{x} \ne 0$, we have $\bar{y}^\top A\bar{x} < 0$. On the other hand, since $\bar{x} \ge 0$ and $A^\top \bar{y} \ge 0$, we have $\bar{y}^\top A\bar{x} \ge 0$. This results in a contradiction.
(S2) Suppose (I) is not solvable. Then, by a simple scaling argument, we see
(I')$Ax \le 0,\ e^\top Ax = -1,\ x \ge 0$ is not solvable either. (I') equivalent to
$$\begin{bmatrix} A & I \\ e^\top A & \mathbf{0}^\top \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix} = \begin{bmatrix} \mathbf{0}^\top \\ -1 \end{bmatrix}, \quad (x, s) \ge \mathbf{0}.$$
By Farkas', $\exists \bar{z} = (\bar{u}, \bar{t}) \in \mathbb{R}^{m+1}$ s.t. $\begin{bmatrix} A^\top & A^\top e \\ I & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{u} \\ \bar{t} \end{bmatrix} \ge \mathbf{0},\ \bar{t} > 0,$
or equivalently, $A^\top(\bar{u} + \bar{t}e) \ge \mathbf{0},\ \bar{u} \ge \mathbf{0},\ \bar{t} > 0$. Now, let $\bar{y} = \bar{u} + \bar{t}e \in \mathbb{R}^m$. Clearly, we have $A^\top \bar{y} \ge 0$. Moreover, since $\bar{u} \ge 0$ and $\bar{t} > 0$, we have $\bar{y} \ge \bar{t}e > 0$. It follows that (II) is solvable, as desired.
(b) $A \in \mathbb{R}^{m\times n},\ b \in \mathbb{R}^m,\ c \in \mathbb{R}^n,\ d \in \mathbb{R}$. $\exists \bar{x} \in \mathbb{R}^n$ satisfying $A\bar{x} \le b$.
(I) $\quad Ax \le b,\ c^\top x > d. \qquad$ (II) $\quad A^\top y = c,\ b^\top y \le d,\ y \ge 0.$
A: (S1) The systems (I) and (II) cannot be simultaneously solvable. If $\bar{x} \in \mathbb{R}^n$ solves (I) and $\bar{y} \in \mathbb{R}^m$ solves (II), then $d < c^\top \bar{x} = \bar{y}^\top A\bar{x} \le b^\top \bar{y} \le d$, which is a contradiction.
(SH) We claim that (I) is solvable iff (I') $Ax - bt \le 0,\ c^\top x - dt > 0,\ t \ge 0$ is solvable. Indeed, if x' solves (I), then $(x', 1)$ solves (I').
Conversely, suppose that (x', t') solves (I').
(i) If $t' > 0$, then it is easy to verify that x'/t' solves (I).
(ii) If $t' = 0$, then $Ax' \le 0$ and $c^\top x' > 0$. Since $A\bar{x} \le b$ by assumption, letting $x'' = \bar{x} + \theta x'$ with $\theta \uparrow \infty$, we have $c^\top x'' = c^\top \bar{x} + \theta c^\top x' > d$ and $Ax'' = A(\bar{x} + \theta x') \le b$. It follows that x'' solves (I).
(S2) Now, note that (I') takes the form
$$\text{(I'')} \quad \begin{bmatrix} A & -b \\ \mathbf{0} & -1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \le \mathbf{0}, \quad \begin{bmatrix} c^\top & -d \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} > 0.$$
Suppose that (I'') is not solvable. By Farkas' lemma, we see that
$$\text{(II')} \quad \begin{bmatrix} A^\top & \mathbf{0} \\ -b^\top & -1 \end{bmatrix} \begin{bmatrix} y \\ s \end{bmatrix} = \begin{bmatrix} c \\ -d \end{bmatrix}, \quad y \ge \mathbf{0}, \quad s \ge 0 \text{ is solvable.}$$
This implies that (II) is solvable, as can be easily verified.
2. Construct a primal-dual pair of linear programs such that both the primal and the dual have a unique optimal solution.
A: There are many possible constructions. For instance, consider the following primal-dual pair of standard-form LP:
$$\text{(P)} \quad \text{minimize } x_1 \qquad \text{subject to } \begin{cases} x_1 + x_2 = 1, \\ x_1 - x_2 = 0, \\ x_1, x_2 \ge 0. \end{cases}$$
$$\text{(D)} \quad \text{maximize } y_1 \qquad \text{subject to } \begin{cases} y_1 + y_2 \le 1, \\ y_1 - y_2 \le 0. \end{cases}$$
3. Let $A \in \mathbb{R}^{m\times n}$ and $c \in \mathbb{R}^n$ be given. Let $v : \mathbb{R}^m \to \mathbb{R}$ be the function defined by $v(b) = \text{minimize } c^\top x \quad \text{subject to } Ax \ge b,\ x \ge 0.$ (1), i.e., $v(b)$ is the optimal value of (1) when RHS of first inequality constraint is b.
(a) Let $b \in \mathbb{R}^m$ be fixed. Find the dual of Problem (1).
(b) Using the result in (a), or otherwise, show that the function $v(\cdot)$ is convex on the set $\{b \in \mathbb{R}^m : v(b) \text{ is finite}\}$.
A: (a) $v(b) = -\max(-c)^\top x \quad$ s.t. $\begin{bmatrix} -A \\ -I \end{bmatrix} x \le \begin{bmatrix} -b \\ 0 \end{bmatrix}$.
Dual: $\max b^\top z \quad$ s.t. $\quad A^\top z + w = c$ (or $A^\top z \le c$), $\quad z, w \ge 0$.
(b) $v(b) = \max b^\top z \quad$ s.t. $A^\top z \le c,\ z \ge 0$. by the LP strong duality theorem. Since $v(\cdot)$ is a pointwise supremum of the collection $\{b \mapsto b^\top z : A^\top z \le c,\ z \ge 0\}$ of linear functions, $v(\cdot)$ is convex.
4. **23Q5.** Let $A \in \mathbb{R}^{p\times d},\ b \in \mathbb{R}^p$ be given. Consider the set: $\mathcal{X} = \{x \in \mathbb{R}^d : Ax = b, x \ge 0\}$. Show that $\mathcal{X}$ is bounded iff there exists $u \in \mathbb{R}^p$ such that $A^\top u > 0$. *Hint:* the set $\mathcal{X}$ is unbounded iff there exists $c \in \mathbb{R}^d$ such that $Ac = 0, c \ge 0, c \ne 0$.
A: Follow the proof of Gordan's Theorem.
21Q4. Let $A \in \mathbb{R}^{m\times n},\ b \in \mathbb{R}^m,\ c \in \mathbb{R}^n$, consider the following two sets:
$S = \{x \in \mathbb{R}^n : Ax \ge b, x \ge 0\}, \quad T = \{y \in \mathbb{R}^m : A^\top y \le c, y \ge 0\}$
(a) Suppose that $T \ne \emptyset, S \ne \emptyset$. Prove that at least one of the above two sets is unbounded. *Hint: notice that if there exists a direction $d \in \mathbb{R}^n$ such that $Ad \ge b, d \ge 0, d \ne 0$, then S must be unbounded.*
(b) Could both of the two sets be unbounded simultaneously?
Farkas: (I) $Ax > 0,\ x \ge 0$. (II) $y^\top A \le 0,\ y \ge 0,\ y \ne 0$. **A:** (a) Suppose Farkas (2) holds: $\exists\, y^*$ s.t. $A^\top y^* \le c,\ y^* \ge 0,\ y^* \ne 0$. Since $T \ne \emptyset$, $\exists\, \bar{y}$ s.t. $A^\top \bar{y} \le c,\ \bar{y} \ge 0,\ \bar{y} \ne 0$. Then consider $y = y^* + \bar{y}$: $A^\top(y^* + \bar{y}) \le c,\ y^* + \bar{y} \ge 0$. Then T is unbounded.
Suppose Farkas' lemma (2) does not hold: $\exists\, x^*$ s.t. $Ax^* > 0$ and $x^* \ge 0$. Since $S \ne 0$, $\exists\, \bar{x}$ s.t. $A\bar{x} \ge b,\ \bar{x} \ge 0$. Then $A\bar{x} + Ax^* \ge b$ and $\bar{x} + x^* \ge 0$. Since $A\bar{x} + Ax^* \ge b, \bar{x} + x^* \ge 0, \bar{x} + x^* \ne 0$, then S is unbounded.
(b) Yes, they can. Let $A = \begin{bmatrix} 0 & 0 \end{bmatrix} \in \mathbb{R}^{1\times 2},\ b = 0,\ c = \begin{bmatrix} 0 \end{bmatrix} \in \mathbb{R}^2$.
Then for any $x \in \mathbb{R}^2_+$, we have $Ax = 0 \ge b = 0$. For any $y \in \mathbb{R}_+$, we have $A^\top y = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \le c = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
Thus, both S and T in this case are unbounded.

5. Farkas: (I) $Ax \geq 0$, $Ax \neq 0$; (II) $A^\top y = 0$, $y > 0$.
A: (S1) Suppose $\bar{x} \in \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}^m$ satisfying (I) and (II). Then, since $\bar{y} > 0$, $A\bar{x} \geq 0$, and $A\bar{x} \neq 0$, we have $\bar{y}^\top A\bar{x} > 0$; since $A^\top \bar{y} = 0$, we have $\bar{y}^\top A\bar{x} = 0$. contradiction.
Now, suppose system (I) no solution. Then the system (I') $Ax \geq 0$, $e^\top Ax = 1$ no solution either. System (I') is equivalent to
$$\begin{bmatrix} A & -A & -I \\ e^\top A & -e^\top A & 0^\top \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (x^+, x^-, s) \geq 0.$$
Hence, by Farkas' lemma, there exists a $\bar{z} = (\bar{u}, \bar{t}) \in \mathbb{R}^{m+1}$ such that
$$\begin{bmatrix} A^\top & A^\top e \\ -A^\top & -A^\top e \\ -I & 0 \end{bmatrix} \begin{bmatrix} \bar{u} \\ \bar{t} \end{bmatrix} \leq \begin{bmatrix} 0 \\ \bar{t} > 0 \end{bmatrix},$$
or equivalently, $A^\top(\bar{u} + \bar{t}e) = 0$, $\bar{u} \geq 0$, $\bar{t} > 0$. Now, let $\bar{y} = \bar{u} + \bar{t}e \in \mathbb{R}^m$. Clearly, we have $A^\top \bar{y} = 0$. Moreover, since $\bar{u} \geq 0$ and $\bar{t} > 0$, we have $\bar{y} \geq \bar{t}e > 0$.

6. Let $A \in \mathbb{R}^{m \times n}$ and $c \in \mathbb{R}^n$ be given. Show that $\{x \in \mathbb{R}^n : Ax \leq 0\} \subseteq \{x \in \mathbb{R}^n : c^\top x \leq 0\}$ if and only if $A^\top y = c$ for some $y \geq 0$.
A: By Farkas' lemma, exactly one of the following systems is solvable:
(I) $A^\top y = c$, $y \geq 0$. (II) $Ax \leq 0$, $c^\top x > 0$.
It follows that (I) is solvable if and only if $c^\top x \leq 0$ whenever $x \in \mathbb{R}^n$ satisfies $Ax \leq 0$.

7. Farkas: (I) $Ax < \mathbf{0}$, $x \geq \mathbf{0}$. (II) $A^\top y \geq \mathbf{0}$, $y \geq \mathbf{0}$, $y \neq \mathbf{0}$.
A: (I') $Ax + s = -e$, $(x, s) \geq \mathbf{0}$.

8. Let $A \in \mathbb{R}^{m \times n}$ and $c \in \mathbb{R}^n$ be given. Define $C = \{x \in \mathbb{R}^n : Ax \geq 0\}$. Suppose that 0 is a basic feasible solution of C. Consider the following LP: $v^* = \min_{x \in C} c^\top x$. Show that $v^* = -\infty$ if and only if there exists a $d \in C \setminus \{0\}$ such that there are $n - 1$ linearly independent active constraints at d and $c^\top d < 0$.
A: Suppose that there exists a $d \in C \setminus \{0\}$ satisfying $c^\top d < 0$. Then, we have $\lambda d \in C$ for any $\lambda > 0$, which implies that $v^* = -\infty$.
Conversely, suppose that $v^* = -\infty$. Let $A \in \mathbb{R}^{m \times n}$ be the matrix whose i-th row is $a_i^\top$, where $i = 1, \ldots, m$. By scaling if necessary, there exists an $\bar{x} \in C$ such that $c^\top \bar{x} = -1$. This implies that the polyhedron $P = \{x \in \mathbb{R}^n : a_i^\top x \geq 0 \text{ for } i = 1, \ldots, m, c^\top x = -1\}$ is non-empty. Since 0 is a basic feasible solution of C, there exist n vectors in the collection $\{a_1, \ldots, a_m\}$ that are linearly independent. Hence, by Theorem 3, P has at least one extreme point, say $d \in \mathbb{R}^n$. Note that there are n linearly independent active constraints at d. Moreover, since $c^\top d = -1$, we have $d \neq 0$. Thus, there are $n - 1$ linearly independent constraints of the form $a_i^\top x \geq 0$ that are active at d.

9. A primal-dual pair of LPs in standard forms such that neither is feasible.
A: $A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $c = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

10. Let $P \in \mathbb{R}^{n \times n}$ be a stochastic matrix; i.e., $P_{ij} \geq 0$ for $i, j \in \{1, \ldots, n\}$ and $Pe = e$. Show the system $P^\top x = x$, $x \geq 0$, $x \neq 0$ is solvable.
A: Consider the LP: $\max\ e^\top x$ s.t. $(I - P)^\top x = \mathbf{0}$ $x \geq 0$.
Dual: $\min\ 0$ s.t. $(P - I)y \geq e$. We claim that (D) is infeasible. Indeed, since P is a stochastic matrix, each entry of the vector Py is a convex combination of the entries of y. In particular, we have $[Py]_i \leq y_{\max}$ for $i = 1, \ldots, n$. However, one of the entries of $y + e$ equals $y_{\max} + 1$. This yields the desired contradiction.
It follows (P) is either infeasible or unbounded. However, $x = 0$ is feasible for (P). Hence, we conclude (P) is unbounded, which implies $P^\top x = x$, $x \geq 0$, $x \neq 0$ is solvable.

11. $\max\ c^\top x$ s.t. $Ax \leq b, x \geq 0$. Dual: $\min\ -b^\top y$ s.t. $A^\top y \leq -c, y \leq 0$
Equivalent to: $\min\ b^\top y$ s.t. $A^\top y \geq c, y \geq 0$.

12. $\min\ c^\top x$ s.t. $Ax \geq c, x \geq 0$. Show if $\bar{x} \in \mathbb{R}^n$ satisfies $A\bar{x} = c$ and $\bar{x} \geq \mathbf{0}$. From 6, dual given by: $\max\ c^\top y$ s.t. $A^\top y \leq c, y \geq 0$.
When $A\bar{x} = c$ and $\bar{x} \geq 0$, $\bar{x}$ is feasible for both P and D, and P and D have the same value. By strong duality, $\bar{x}$ is optimal.

13. Reformulate: $\min\ \|Ax - b\|_2^2$ s.t. $\|x\|_0 \leq K$, $x \in \mathbb{R}^n$. We are given a constant $M > 0$ such that $\|x^*\|_0 \leq M$ for some optimal solution x^*. **A:** $\min\ \|Ax - b\|_2^2$ s.t. $\sum_{i=1}^n y_i \leq K$, $x_i \leq My_i$ for $i = 1, \ldots, n$,
$x_i \geq -My_i$ for $i = 1, \ldots, n$, $y_i \in \{0, 1\}$ for $i = 1, \ldots, n$.

14. Let $P \subseteq \mathbb{R}^n$ be a non-empty polyhedron. Suppose for $i = 1, \ldots, n$, we either have constraint $x_i \geq 0$ or constraint $x_i \leq 0$ in description of P. Does P have at least 1 vertex? **A:** Yes. By assumption, the polyhedron P contains the constraints $\begin{cases} x_i \geq 0 & \text{for } i \in I, \\ x_i \leq 0 & \text{for } i \notin I, \end{cases}$ where $I \subseteq \{1, \ldots, n\}$. We claim that P does not contain a line, which would then imply the desired conclusion. Suppose that this is not the case. Then, there exist $x_0 \in P$ and $d \neq 0$ such that $x_0 + \alpha d \in P$ for all $\alpha \in \mathbb{R}$. Let $j \in \{1, \ldots, n\}$ be such that $d_j \neq 0$. If $j \in I$, then $(x_0 + \alpha d)_j < 0$ as $\alpha \to -\infty$, which contradicts the hypothesis that $(x_0 + \alpha d)_j \geq 0$ for all $\alpha \in \mathbb{R}$. Also a similar contradiction for the case where $j \notin I$.

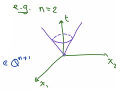
5 Conic Linear Programming

5.1 Introduction

The relation $\geq$ defines a *partial order* on vectors in $\mathbb{R}^n$; i.e., it satisfies
(a) (Reflexivity) $u \geq u$ for all $u \in \mathbb{R}^n$;
(b) (Anti-Symmetry) $u \geq v$ and $v \geq u$ imply $u = v$ for all $u, v \in \mathbb{R}^n$;
(c) (Transitivity) $u \geq v$ and $v \geq w$ imply $u \geq w$ for all $u, v, w \in \mathbb{R}^n$.
The relation $\geq$ is *compatible with linear operations*; i.e., it satisfies
(d) (Homogeneity) for any $u, v \in \mathbb{R}^n$ and $\alpha \geq 0$, if $u \geq v$, then $\alpha u \geq \alpha v$;
(e) (Additivity) for any $u, v, w, z \in \mathbb{R}^n$, if $u \geq v$ and $w \geq z$, then $u + w \geq v + z$.
Every good relation $\succeq$ on E induces a **pointed cone** $K = \{u \in E : u \succeq \mathbf{0}\}$ with $\mathbf{0} \in K$:
1. K is *non-empty and closed under addition*; i.e., $u + v \in K\ \forall u, v \in K$.
2. K is a *cone*; i.e., for any $u \in K$ and $\alpha \geq 0$, we have $\alpha u \in K$.
3. K is *pointed*; i.e., if $u \in K$ and $-u \in K$, then $u = 0$.
The first property follows from (a) (which implies that $0 \in K$) and (e); the second follows from (d). The third: observe $u \succeq u$ by (a) with $-u \succeq 0$ and (e) implies that $0 \succeq u$. Since $u \succeq 0$, it follows from (b) that $u = 0$.
Note that a pointed cone K is automatically convex. To prove this, let $u, v \in K$ and $\alpha \in (0, 1)$. Then, since K is a cone, we have $\alpha u, (1 - \alpha)v \in K$. Since K is closed under addition, we conclude that $\alpha u + (1 - \alpha)v \in K$
The converse is also true; i.e., given an arbitrary pointed cone $K \subseteq E$ with $0 \in K$, we can define a good relation on E. $u \succeq_K v \iff u - v \in K$. By definition, we have $u \succeq_K v$ iff $u - v \succeq_K 0$. Now, we claim $\succeq_K$ is good:
(a) (Reflexivity) Since $\mathbf{0} \in K$, we see that for any $u \in E$, we have $u - u \in K$; i.e., $u \succeq_K u$. (b) (Anti-Symmetry) If $u - v \in K$ and $v - u \in K$, then by the pointedness of K, we have $u - v = \mathbf{0}$; i.e., $u = v$. (c) (Transitivity) If $u - v \in K$ and $v - w \in K$, then by the addition property, we have $u - w \in K$; i.e., $u \succeq_K w$. (d) (Homogeneity) Suppose that $u - v \in K$ and $\alpha \geq 0$. By the conic property, we have $\alpha(u - v) \in K$, which implies that $\alpha u \succeq_K \alpha v$. The case where $\alpha = 0$ trivially follows from reflexivity. (e) (Additivity) Suppose that $u - v \in K$ and $w - z \in K$. By the addition property, we have $u + w - (v + z) \in K$; i.e., $u + w \succeq_K v + z$.

Example 1 (Representative *Closed* Pointed Cones)
1. Non-Negative Orthant. $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x \geq 0\}$. (a pointed cone in $\mathbb{R}^n$ equipped with the usual inner product)
Good relation: For $u, v \in \mathbb{R}^n$, $u \geq v$ iff $u_i \geq v_i\ \forall i = 1, \ldots, n$.
2. Lorentz Cone (SOC). $Q^{n+1} = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t \geq \|x\|_2\}$.
(a pointed cone in $\mathbb{R}^{n+1}$ equipped with the usual inner product)
Good relation: For $(s, u), (t, v) \in \mathbb{R} \times \mathbb{R}^n$, $(s, u) \succeq_{Q^{n+1}} (t, v)$ iff $s - t \geq \|u - v\|_2$.

Proof: closure under addition:
$(t, x) \in Q^{n+1}$ $\iff$ $t \geq \|x\|_2$
$(s, y) \in Q^{n+1}$ $\quad$ $s \geq \|y\|_2$
$\Rightarrow$ $t + s \geq \|x\|_2 + \|y\|_2 \geq \|x + y\|_2$ $\Rightarrow$ $(t + s, x + y) \in Q^{n+1}$
(triangle inequality)

e.g. n=2


3. Positive Semidefinite Cone. $S_+^n = \{X \in S^n : u^\top Xu \geq 0 \forall u \in \mathbb{R}^n\}$ $= \{X \in S^n : \lambda_{\min}(X) \geq 0\}$.
(a pointed cone in S^n of $n \times n$ symmetric matrices equipped with Frobenius inner product $X \cdot Y = \text{tr}(X^\top Y) = \text{tr}(XY) = \sum_{i=1}^n \sum_{j=1}^n X_{ij} Y_{ij}$)
(note that S^n can be identified with $\mathbb{R}^{n(n+1)/2}$).
Good relation: positive semidefinite ordering: For $X, Y \in S^n$, we have $X \succeq Y$ iff $X - Y$ is positive semidefinite (denoted by $X - Y \succeq \mathbf{0}$).
4. Zero Cone. $K = \{0\}$.
All cones in Example 1 are closed and have non-empty interiors, consequences: First, if $\{u^i\}$, $\{v^i\}$ are sequences in E such that
$u^i \succeq_K v^i$ for $i = 1, 2, \ldots$; $u^i \to u \in E$; $v^i \to v \in E$, then $u \succeq_K v$.
Second, if the pointed cone K has a non-empty interior, then define a strict relation $\succ_K$ via $u \succ_K v \iff u - v \in \text{int}(K)$.
Proposition 1. Let $E_1, \ldots, E_n$ be finite-dimensional Euclidean spaces and $K_i \subseteq E_i$ be closed pointed cones with non-empty interiors, where $i = 1, \ldots, n$. Then, the set
$K \equiv K_1 \times \ldots \times K_n = \{(x_1, \ldots, x_n) \in E_1 \times \ldots \times E_n : x_i \in K_i \text{ for } i = 1, \ldots, n\}$
is a closed pointed cone with non-empty interior.

5.2 Conic Linear Programming

Let E be a finite-dimensional Euclidean space equipped with an inner product $\bullet$ and a good relation $\succeq$. we define the **standard form Conic Linear Programming (CLP) problem** as follows:
$v_p^* = \inf c \bullet x$ s.t. $a_i \bullet x = b_i$ for $i = 1, \ldots, m$, $x \succeq_K \mathbf{0}$ (P).
Define the **dual cone** of the cone K as $K^* = \{w \in E \mid x \bullet w \geq 0 \forall x \in K\}$.
The dual of (P) can be given by:
$v_d^* = \sup b^\top y$ s.t. $c - \sum_{i=1}^m y_i a_i \in K^*$, $y \in \mathbb{R}^m$ (D), or
$v_d^* = \sup b^\top y$ s.t. $\sum_{i=1}^m y_i a_i + s = c$, $y \in \mathbb{R}^m$, $s \succeq_{K^*} \mathbf{0}$ (D).
Proposition 2 Let $K \subseteq E$ be a non-empty set. Then, the following hold:
(a) The set K^* is a closed convex cone, regardless of what $K (\neq \emptyset)$ is.
(b) If K is a closed convex cone, then so is K^*. Moreover, $(K^*)^* = K$.
(c) If K has a non-empty interior, then K^* is pointed.
(d) If K is a closed pointed cone, then K^* has a non-empty interior.
Proof (b) It is clear from the definition that $K \subseteq (K^*)^*$.
To establish the converse, let $v \in (K^*)^*$ be arbitrary. If $v \notin K$, then by the separation theorem, $\exists y \in \mathbb{R}^n$ s.t. $\inf_{x \in K} y^\top x > y^\top v$. We claim that $\theta^* = \inf_{x \in K} y^\top x = 0$: Clearly, we have $\theta^* \leq 0$ since $0 \in K$. Now, if $\theta^* < 0$, then $\exists x' \in K$ s.t. $0 > y^\top x' > y^\top v$. However, since $\alpha x' \in K$ for all $\alpha > 0$, we see that $\alpha y^\top x' > y^\top v$ for all $\alpha \geq 1$, which is impossible. Thus, the claim is established. In particular, this shows that $y \in K^*$. However, we then have the inequality $0 > y^\top v$, which contradicts the fact that $v \in (K^*)^*$. Hence, we conclude that $v \in K$.
(c) Suppose that K^* is not pointed. Then, $\exists w \in K^*$ s.t. $w \neq 0$ and $x \bullet w = 0$ for all $x \in K$. This implies that K is a subset of the hyperplane $H(w, 0) = \{x \in E : w \bullet x = 0\}$, which shows that $\text{int}(K) = \emptyset$.
(d) Suppose that $\text{int}(K^*) = \emptyset$. Then, there exists a hyperplane $H(s, 0) = \{w \in E : s \bullet w = 0\}$ with $s \neq 0$ s.t. $K^* \subseteq H(s, 0)$. Since K is a closed convex cone by assumption, using the result in (b), we compute $K = (K^*)^* = \{x \in E : x \bullet w \geq 0 \text{ for all } w \in K^*\}$
$\supseteq \{x \in E : x \bullet w \geq 0 \text{ for all } w \in H(s, 0)\} = \{\lambda s : \lambda \in \mathbb{R}\}$.
This shows that K is not pointed. □
Corollary 1 Let $K \subseteq E$ be a closed pointed cone with non-empty interior. Then, so is the dual cone $K^* \subseteq E$.
Proposition 3 Let $E_1, \ldots, E_n$ be finite-dimensional Euclidean spaces equipped with the inner products $\bullet_1, \ldots, \bullet_n$, respectively. Let $E = E_1 \times \ldots \times E_n$ and define the inner product $\bullet$ on E by
$u \bullet v = \sum_{i=1}^n u_i \bullet_i v_i$ where $u_i, v_i \in E_i$, for $i = 1, \ldots, n$.
Suppose that $K_i \subseteq E_i$ (where $i = 1, \ldots, n$) are closed pointed cones with non-empty interiors and $K = K_1 \times \ldots \times K_n$. Then, the dual cone K^* is $K^* = K_1^* \times \ldots \times K_n^*$ and is a closed pointed cone with non-empty interior.
Observation In (D), the objective function is linear, and the map $\mathbb{R}^m \ni y \mapsto c - \sum_{i=1}^m a_i y_i \in E$ is affine:
$M(\alpha y + (1 - \alpha)z) = \alpha M(y) + (1 - \alpha)M(z)$.
Observation $(\mathbb{R}_+^n)^* = \mathbb{R}_+^n$; $(Q^{n+1})^* = Q^{n+1}$; $(\mathbb{S}_+)^* = \mathbb{S}_+$.
Proof (1) On one hand, we have $\mathbb{R}_+^n \subseteq (\mathbb{R}_+^n)^*$ because $y^\top x \geq 0$ if $x, y \geq 0$. On the other hand, suppose that $y \in (\mathbb{R}_+^n)^*$. Then, we have $x^\top y \geq 0$ for all $x \in \mathbb{R}_+^n$. In particular, we have $e_i^\top y = y_i \geq 0$ for $i = 1, \ldots, n$, where $e_i \in \mathbb{R}^n$ is the i-th standard basis vector. This shows that $y \in \mathbb{R}_+^n$, as desired.
(2) $(Q^{n+1})^* = \{(s, y) \in \mathbb{R} \times \mathbb{R}^n : st + x^\top y \geq 0, \forall (t, x) \in Q^{n+1}\}$.
Prove $Q^{n+1} \subseteq (Q^{n+1})^*$: Suppose $(s, y) \in Q^{n+1}$, i.e., $s \geq \|y\|_2$. Then, $\forall (t, x) \in Q^{n+1}$, $st \geq \|x\|_2 \|y\|_2 \geq -x^\top y$ (by Cauchy-Schwarz). This implies $(s, y) \in (Q^{n+1})^*$. Thus, $Q^{n+1} \subseteq (Q^{n+1})^*$.
Prove $(Q^{n+1})^* \subseteq Q^{n+1}$: Suppose $(s, y) \in (Q^{n+1})^*$. Since $(1, \mathbf{0}) \in Q^{n+1}$, we have $s \geq \mathbf{0}$. If $y = 0$, then automatically we have $(s, y) \in Q^{n+1}$.
If $y \neq \mathbf{0}$, $\left(s, -\frac{sy}{\|y\|_2}\right) \in Q^{n+1} \Rightarrow 0 \leq (s, y)^\top \left(s, -\frac{sy}{\|y\|_2}\right) = s^2 - s\|y\|_2$. If $s > 0$, this is equivalent to $s \geq \|y\|_2$, i.e., $(s, y) \in Q^{n+1}$.
If $s = 0$, $(s, y) \in (Q^{n+1})^* \Rightarrow x^\top y \geq 0 (\forall x \in \mathbb{R}^n) \Rightarrow y = \mathbf{0} \Rightarrow (s, y) = (0, \mathbf{0}) \in Q^{n+1}$.
(3) (HW) By definition, $(S_+^n)^* = \{Y \in S^n : X \bullet Y \geq 0 \text{ for all } X \in S_+^n\}$.
Suppose that $Y \in S_+^n$. Let $Y = U\Sigma U^\top$ be its spectral decomposition. Then, for any $X \in S_+^n$, we have $U^\top XU \in S_+^n$ and hence
$X \bullet Y = (U^\top XU) \bullet \Sigma = \sum_{i=1}^n (U^\top XU)_{ii} \Sigma_{ii} \geq 0$. It follows $Y \in (S_+^n)^*$.
Conversely, suppose that $Y \in (S_+^n)^*$. Let $Y = U\Sigma U^\top$ be its spectral decomposition. Since $X_i = Ue_ie_i^\top U \in S_+^n$ for $i = 1, \ldots, n$, we have $0 \leq X_i \bullet Y = \Sigma_{ii}$ for $i = 1, \ldots, n$. This shows that all the eigenvalues of Y are non-negative, which implies that $Y \in S_+^n$, as desired.□
By Corollary 1, if K is a closed pointed cone with non-empty interior, then so is K^*. In this case, (P) and (D) are of the same nature: both optimizing a linear function over a set defined by linear equality constraints and a conic constraint that is associated with a closed pointed cone with non-empty interior.
Example 2 (Representative CLP Problems)
1. Linear Programming (LP). By taking $E = \mathbb{R}^n$, $K = \mathbb{R}_+^n$, and $u \bullet v = u^\top v$ for $u, v \in E$, Problem (P) becomes
$\inf c^\top x$ s.t. $a_i^\top x = b_i$ for $i = 1, \ldots, m$, $x \in \mathbb{R}_+^n$,
$K^* = (\mathbb{R}_+^n)^* = \mathbb{R}_+^1$. Problem (D) becomes
$\sup b^\top y$ s.t. $\sum_{i=1}^m y_i a_i + s = c$, $y \in \mathbb{R}^m$, $s \in \mathbb{R}_+^n$,
which are LP primal and dual forms.
2. Second-Order Cone Programming (SOCP) Let $E = \mathbb{R}^{n+1}$, $K = Q^{n+1}$, and $u \bullet v = u^\top v$ for $u, v \in E$. Then, Problem (P) becomes
$\inf c^\top x$ s.t. $a_i^\top x = b_i$ for $i = 1, \ldots, m$, $x \in Q^{n+1}$, **(SOCP)**
which is an SOCP in standard primal form.
$(Q^{n+1})^* = Q^{n+1}$. Thus, Problem (D) becomes
$\sup b^\top y$ s.t. $\sum_{i=1}^m y_i a_i + s = c$, $y \in \mathbb{R}^m$, $s \in Q^{n+1}$, **(SOCD)**
which is an SOCP in standard dual form.
Explicitly: Let $a_i = (u_i, a_{i,1}, \ldots, a_{i,n}) \in \mathbb{R}^{n+1}$ and $c = (v, d) \in \mathbb{R}^{n+1}$ with $d \in \mathbb{R}^n$. Then, we have $\sum_{i=1}^m y_i a_i = (u^\top y, A^\top y)$, where $A \in \mathbb{R}^{m \times n}$ is the matrix whose i-th row contains the entries $a_{i,1}, \ldots, a_{i,n}$.
It follows that the constraint $s = c - \sum_{i=1}^m y_i a_i \in Q^{n+1}$ is equivalent to
$(v - u^\top y, d - A^\top y) \in Q^{n+1}$, i.e., $v - u^\top y \geq \|d - A^\top y\|_2$,
which implies Problem (SOCP) takes the form
$\sup b^\top y$ s.t. $(v - u^\top y, d - A^\top y) \in Q^{n+1}$. (SOCD')
In other words, the problem of optimizing a linear function subject to the constraint that the image of an affine map belongs to a SOC is an SOCP.
Note: Matrix form: $A = \begin{bmatrix} -a_1^\top- \\ \ldots \\ -a_m^\top- \end{bmatrix}$, $Ax = \begin{bmatrix} a_1^\top x \\ \ldots \\ a_m^\top x \end{bmatrix}$, $(Ax)_i = a_i^\top x$;
$A^\top y = [a_1, \ldots, a_m] \begin{bmatrix} y_1 \\ \ldots \\ y_m \end{bmatrix} = \begin{bmatrix} y_1 a_{11} + \ldots + y_m a_{m1} \\ \ldots \\ y_1 a_{1n} + \ldots + y_m a_{mn} \end{bmatrix} = \sum_{i=1}^m y_i a_i$.
A n extension of Problem (SOCD) is to allow multiple SOC constraints; i.e.,
$\sup b^\top y$
s.t. $(v_j - (u^j)^\top y, d^j - (A^j)^\top y) \in Q_j^{n_j+1}$ for $j = 1, \ldots, p$ (6),
where $u^j \in \mathbb{R}^m$, $v_j \in \mathbb{R}$, $A^j \in \mathbb{R}^{m \times n_j}$, and $d^j \in \mathbb{R}^{n_j}$ for $i = 1, \ldots, p$.
This can also be viewed as an SOCP, as it can be put into the form
$\sup b^\top y$
s.t. $\sum_{i=1}^m y_i \bar{a}_i + \bar{s} = \bar{c}$, $y \in \mathbb{R}^m$, $\bar{s} \in Q_1^{n_1+1} \times \cdots \times Q_p^{n_p+1}$.
Here, $\bar{c} = (c^1, \ldots, c^p)$, $\bar{s} = (s^1, \ldots, s^p)$, and $\bar{a}_i = (a_i^1, \ldots, a_i^p)$ are vectors in $\mathbb{R}^{n_1+1} \times \cdots \times \mathbb{R}^{n_p+1}$, with $c^j = (v_j, d^j) \in \mathbb{R}^{n_j+1}$ and $a_i^j = (u_i^j, a_{i,1}^j, \ldots, a_{i,n_j}^j) \in \mathbb{R}^{n_j+1}$.
Using (6), it is immediate that the class of SOCPs includes the class of LPs as a special case: the standard form LP can be formulated as the following SOCP:
$\inf c^\top x$ s.t. $\|Ax - b\|_2 \leq 0$, $\|0x - 0\|_2 \leq e_i^\top x$ for $i = 1, \ldots, n$.
3. Semidefinite Programming (SDP) By taking $E = S^n$, $K = S_+^n$,
$X \bullet Y = \text{tr}(X^\top Y)$ for $X, Y \in E$, Problem (P) becomes
$\inf C \bullet X$ s.t. $A_i \bullet X = b_i$ for $i = 1, \ldots, m$, $X \in S_+^n$, **(SDP)**
$(S_+^n)^* = S_+^n$, Problem (D) becomes
$\sup b^\top y$ s.t. $\sum_{i=1}^m y_i A_i + S = C$, $y \in \mathbb{R}^m$, ; $S \in S_+^n$. **(SDD)**
Theorem 1 (CLP Weak Duality) Let $\bar{x} \in K$ be feasible for (P) and $(\bar{y}, \bar{s}) \in \mathbb{R}^m \times K^*$ be feasible for (D). Then, $b^\top \bar{y} \leq c \bullet \bar{x}$.
Farkas-type lemma for the following conic linear systems: (I) $a_i \bullet x = b_i$ for $i = 1, \ldots, m$, $x \in K$.
(II) $-\sum_{i=1}^m y_i a_i \in K^*$, $b^\top y > 0$.
where E is a finite-dimensional Euclidean space equipped with inner product $\bullet$, $K \subseteq E$ is a closed pointed cone with non-empty interior.
It is straightforward to show that (I) and (II) cannot both have solutions.
Proof: Suppose that $\bar{x} \in E$ is a solution to (I) and $\bar{y} \in \mathbb{R}^m$ is a solution to (II).
Then, we have $0 < b^\top \bar{y} = \sum_{i=1}^m y_i (a_i \bullet x) = -\left(-\sum_{i=1}^m y_i a_i\right) \bullet x \leq 0$,
which is a contradiction. □
Example 3 ("Failure" of the Conic Farkas Lemma) Let $E = S^2$ and $K = S_+^2$. Define $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.
Consider the following systems: (I) $A_1 \bullet X = b_1$, $A_2 \bullet X = b_2$, $X \in S_+^2$.
(II) $-(y_1 A_1 + y_2 A_2) \in S_+^2$, $b^\top y > 0$.
Observe that (I) is equivalent to $X_{11} = 0$, $X_{12} = 1$, $X \in S_+^2$
and (II) is equivalent to $-\begin{bmatrix} y_1 & y_2 \\ y_2 & 0 \end{bmatrix} \in S_+^2$, $y_2 > 0$, both insolvable.
Why a Farkas-type lemma need not hold for the systems (I) and (II):
(1) In the proof of Farkas' lemma for linear systems, we show that the set $S = \{Ax \in \mathbb{R}^m : x \in \mathbb{R}_+^n\}$ is closed and convex, so that we can apply the separation theorem to $\{b\}$ and S if system (A) is insolvable.
(2) For a general closed pointed cone K, the set $S' = \{(a_1 \bullet x, \ldots, a_m \bullet x) \in \mathbb{R}^m : x \in K\}$ need not be closed. In the setting of Example 3, we have $S' = \{(A_1 \bullet X, A_2 \bullet X) : X \in S_+^2\} = \{(X_{11}, 2X_{12}) : X \in S_+^2\}$.
Since $X \in S_+^2$ iff all of its principal minors are non-negative,
$S' = \{(0, 0)\} \cup \{(x, y) : x > 0, y \in \mathbb{R}\} \subset \mathbb{R}^2$, clearly not closed.
Note: We used the fact: *Principal minor characterization:*
1. If $X \succeq 0$, then $X_{ii} \geq 0\ \forall i$; 2. If $X \succeq 0$, $X_{ii} = 0 \Rightarrow X_{ij} = 0\ \forall j$;
3. $S = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \succeq 0 \iff a, c \geq 0; \det(S) \geq 0$
Theorem 2 (Conic Farkas' Lemma) Let E be a finite-dimensional Euclidean space equipped with an inner product $\bullet$, $K \subseteq E$ be a closed pointed cone with non-empty interior, and $a_1, \ldots, a_m \in E$ and $b \in \mathbb{R}^m$ be given vectors. Suppose that the Slater condition holds; i.e., there exists a $\bar{y} \in \mathbb{R}^m$ such that $-\sum_{i=1}^m \bar{y}_i a_i \in \text{int}(K^*)$. Note that $\bar{y}$ need not satisfy $b^\top \bar{y} > 0$. Then, exactly one of the systems (I) and (II) has a solution.
Theorem 3 (CLP Strong Duality)
(a) Suppose that (P) is bounded below and strictly feasible; i.e., there exists a feasible solution $\bar{x}$ to (P) such that $\bar{x} \in \text{int}(K)$. Then, $v_p^* = v_d^*$. Moreover, there exists a feasible solution $(\bar{y}, \bar{s})$ to (D) such that $b^\top \bar{y} = v_p^* = v_d^*$; i.e., the common optimal value is attained by some dual feasible solution.
(b) Suppose that (D) is bounded above and strictly feasible; i.e., there exists a feasible solution $(\bar{y}, \bar{s})$ to (D) such that $\bar{s} = c - \sum_{i=1}^m \bar{y}_i a_i \in \text{int}(K^*)$. Then, $v_p^* = v_d^*$. Moreover, there exists a feasible solution $\bar{x}$ to (P) such that $c \bullet \bar{x} = v_p^* = v_d^*$; i.e., the common optimal value is attained by some primal feasible solution.
(c) Suppose that either (P) or (D) is bounded and strictly feasible. Then, given a feasible solution $\bar{x}$ to (P) and a feasible solution $(\bar{y}, \bar{s})$ to (D), the following are equivalent:
- $\bar{x}$ and $(\bar{y}, \bar{s})$ are optimal for (P) and (D), respectively.
- The duality gap is zero; i.e., $c \bullet \bar{x} = b^\top \bar{y}$.
- Complementary slackness: i.e., $\bar{x} \bullet \bar{s} = \bar{x} \bullet (c - \sum_{i=1}^m \bar{y}_i a_i) = 0$.

Compare with LP: whenever one of (P) or (D) is *bounded* and *feasible*, then (i) the optimal value v_p^* of (P) and the optimal value v_d^* of (D) are equal, and (ii) there exists a primal feasible solution $\bar{x}$ and a dual feasible solution $(\bar{y}, \bar{s})$

such that $c^\top \bar{x} = v_p^* = v_d^* = b^\top \bar{y}$.

Primal and dual attainment of the common optimal value is implied by the boundedness and feasibility of **either** the primal or the dual LP problem.

Example 4 (Pathologies in Conic Duality)

1. Both the primal problem (P) and the dual problem (D) are both bounded and feasible, but the duality gap is non-zero. Consider the SDP

$$\inf\ X_{12} \quad \text{s.t. } X = \begin{bmatrix} 0 & X_{12} & 0 \\ X_{12} & X_{22} & 0 \\ 0 & 0 & 1 + X_{12} \end{bmatrix} \in S^3_+. \quad (10)$$

It is a routine exercise to show that the dual of (10) is given by

$$\sup\ y_4 \quad \text{s.t. } S = \begin{bmatrix} -y_1 & \frac{1+y_4}{2} & -\frac{y_2}{2} \\ \frac{1+y_4}{2} & 0 & -\frac{y_3}{2} \\ -\frac{y_2}{2} & -\frac{y_3}{2} & -y_4 \end{bmatrix} \in S^3_+. \quad (11)$$

Since $X \in S^3_+$, we must have $X_{12} = 0$, which implies that the optimal value of (10) is 0. Similarly, since $S \in S^3_+$, we must have $(1 + y_4)/2 = 0$, or equivalently, $y_4 = -1$. Hence, the optimal value of (11) is -1.

X rank at most 2, violates Slater (need pd: full rank), but has attainment.

2. The primal problem (P) is bounded below and strictly feasible, but the optimal value is not attained by any primal feasible solution (attained by (D)).

Consider the SOCP: $\inf\ x_1 \quad \text{s.t. } (x_1 + x_2, 1, x_1 - x_2) \in Q^3. \quad (12)$

Note that the constraint in (12) is equivalent to $x_1 + x_2 \geq \sqrt{1 + (x_1 - x_2)^2}$, which in turn is equivalent to $4x_1x_2 \geq 1, \quad x_1 + x_2 > 0. \quad (13)$

By (13), we see that the optimal value of (12) is bounded below by 0. Moreover, for $x_1 = x_2 = 1$, we have $(2, 1, 0) \in \text{int}(Q^3)$, which implies that (12) is strictly feasible. Now, by setting $x_1 = 1/(4x_2)$ and letting $x_2 \to \infty$, we see that the optimal value of (12) is 0. However, such an optimal value is not attained by any feasible solution to (12). Dual of (12):

$\sup\ -y_2 \quad \text{s.t. } y_1 + y_3 = 1, \ y_1 - y_3 = 0, \ (y_1, y_2, y_3) \in Q^3. \quad (14)$

The feasible set of (14) is $\{(1/2, 0, 1/2)\}$, which shows that the optimal value of (14) is 0. However, it is clear that (14) is not strictly feasible i.e., violates Slater condition, that's why not attained by (P).

5.3 Some Applications of Conic Linear Programming

5.3.1 Robust Linear Programming

Consider the LP $\min\ \bar{c}^\top x \quad \text{s.t. } \hat{A}x \leq \hat{b} \quad (1)$,

where $\hat{A} \in \mathbb{R}^{m\times n}$, $\hat{b} \in \mathbb{R}^m$, and $\bar{c} \in \mathbb{R}^n$ are given. The data of the above LP $\hat{A}$ and $\hat{b}$ are uncertain. In the robust optimization setting, we assume the uncertain data lie in some given uncertainty set $\mathcal{U}$. Rewrite (1) as:

$\min\ c^\top z \quad \text{s.t. } Az \leq 0, \ z_{n+1} = -1, \quad (2)$

where $A = [\hat{A}\ \hat{b}] \in \mathbb{R}^{m\times(n+1)}$, $c = (\bar{c}, 0) \in \mathbb{R}^{n+1}$, and $z \in \mathbb{R}^{n+1}$.

Now, suppose that each row $a_i \in \mathbb{R}^{n+1}$ of the matrix A lies in an ellipsoidal region $\mathcal{U}_i$ whose center $u_i \in \mathbb{R}^{n+1}$ is given (here, $i = 1, \ldots, m$).

$a_i \in \mathcal{U}_i = \left\{x \in \mathbb{R}^{n+1} : x = u_i + B_i v, \|v\|_2 \leq 1\right\}$ for $i = 1, \ldots, m$,

where a_i is the i-th row of A, $u_i \in \mathbb{R}^{n+1}$ is the center of the ellipsoid $\mathcal{U}_i$, and B_i is some $(n+1)\times(n+1)$ positive semidefinite matrix. Then, (2) is:

$\min\ c^\top z \quad \text{s.t. } Az \leq 0 \, \forall a_i \in \mathcal{U}_i, \ i = 1, \ldots, m, \ z_{n+1} = -1. \quad (3)$

We claim (3) is equivalent to an SOCP problem: observe that $a_i^\top z \leq 0$ for all $a_i \in \mathcal{U}_i$ iff $0 \geq \max_{v\in\mathbb{R}^{n+1}:\|v\|_2\leq 1}\{(u_i + B_i v)^\top z\} = u_i^\top z + \|B_i z\|_2$,

where $i = 1, \ldots, m$. Hence, we conclude that (3) is equivalent to

$\min\ c^\top z \quad \text{s.t. } \|B_i z\|_2 \leq -u_i^\top z$ for $i = 1, \ldots, m, \ z_{n+1} = -1$.

5.3.2 Chance Constrained Linear Programming

$\min\ c^\top x \quad \text{s.t. } a^\top x \leq b, \ x \in P, \quad (4)$

where $a, c \in \mathbb{R}^n$, $b \in \mathbb{R}$ are given, and $P \subseteq \mathbb{R}^n$ is a given polyhedron.

Suppose P is deterministic, but data a, b are randomly affinely perturbed; i.e., $a = a^0 + \sum_{i=1}^l \epsilon_i a^i, \quad b = b^0 + \sum_{i=1}^l \epsilon_i b_i$,

where $a^0, a^1, \ldots, a^l \in \mathbb{R}^n$ and $b^0, b^1, \ldots, b^l \in \mathbb{R}$ are given, and $\epsilon_1, \ldots, \epsilon_l$ are i.i.d. mean zero r.v.s supported on $[-1, 1]$. Then, for any given tolerance parameter $\delta \in (0, 1)$, we can formulate the following:

$\min\ c^\top x \quad \text{s.t. } \Pr(a^\top x > b) \leq \delta(\dagger), \ x \in P. \quad (5)$

In other words, a solution $\bar{x} \in P$ is feasible for (5) if it only violates the constraint $a^\top x \leq b$ with probability at most δ. The constraint $(\dagger)$ is known as a *chance constraint*. Note that when $\delta = 0$, (5) reduces to a robust linear optimization problem. Moreover, if $\bar{x} \in \mathbb{R}^n$ is feasible for (5) at some tolerance level $\bar{\delta} \geq 0$, then it is also feasible for (5) at any $\delta \geq \bar{\delta}$.

Indeed, even when the distributions of $\epsilon_1, \ldots, \epsilon_l$ are very simple, the feasible set defined by the chance constraint $(\dagger)$ can be non-convex. One way to tackle this problem is to replace the chance constraint by its *safe tractable approximation*; i.e., a system of deterministic constraints $\mathcal{H}$ such that (i) $\bar{x} \in \mathbb{R}^n$ is feasible for $(\dagger)$ whenever it is feasible for $\mathcal{H}$ (safe approximation), and (ii) the constraints in $\mathcal{H}$ are efficiently computable (tractability).

We first observe that $(\dagger)$ is equivalent to the following system of constraints:

$\Pr\left(y_0 + \sum_{i=1}^l \epsilon_i y_i > 0\right) \leq \delta, \quad (6)$

$y_i = (a^i)^\top x - b_i \quad$ for $i = 0, 1, \ldots, l. \quad (7)$

Since $\epsilon_1, \ldots, \epsilon_l$ are i.i.d. mean zero random variables supported on $[-1, 1]$, by Hoeffding's inequality, we have

$\Pr\left(\sum_{i=1}^l \epsilon_i y_i > t\right) \leq \exp\left(-\frac{t^2}{2\sum_{i=1}^l y_i^2}\right)$ for any $t > 0$.

It follows that when $-y_0 \geq \sqrt{(2\ln\frac{1}{\delta})\sum_{i=1}^l y_i^2}, \quad (8)$ we have

$\Pr\left(y_0 + \sum_{i=1}^l \epsilon_i y_i > 0\right) \leq \exp\left(-\frac{y_0^2}{2\sum_{i=1}^l y_i^2}\right) \leq \delta.$

In other words, (8) is a sufficient condition for (6) to hold. The upshot of (8) is that it is a SOC constraint. Hence, we conclude that constraints (7) and (8) together serve as a safe tractable approximation of the chance constraint $(\dagger)$.

Constraint (8) is equivalent to the following robust constraint:

$d^\top y \leq 0$ for all $d \in \mathcal{U}$, where $\mathcal{U}$ is the ellipsoidal uncertainty set given by $\mathcal{U} = \left\{x \in \mathbb{R}^{l+1} : x = e_1 + Bv, \|v\|_2 \leq 1\right\}$, and $B \in \mathbb{R}^{(l+1)\times(l+1)}$ is given by $B = \sqrt{2\ln\frac{1}{\delta}} \cdot \begin{bmatrix} 0 & \mathbf{0}^\top \\ \mathbf{0} & I \end{bmatrix}$.

In other words, we are using the following robust optimization problem

$\min\ c^\top x \quad \text{s.t. } \sum_{i=0}^l d_i\left((a^i)^\top x - b_i\right) \leq 0$ for all $d \in \mathcal{U}, \ x \in P$

as a safe tractable approximation of the problem (5).

5.3.3 Quadratically Constrained Quadratic Optimization

$\min\ x^\top Cx \quad \text{s.t. } x^\top A_i x \geq b_i$ for $i = 1, \ldots, m, \quad (9)$

where $C, A_1, \ldots, A_m \in S^n$ are given. We first observe for any $C \in S^n$, $x^\top Cx = \text{tr}(x^\top Cx) = \text{tr}(Cxx^\top) = C \bullet xx^\top$. Hence, (9) is equivalent to

$\min\ C \bullet xx^\top \quad \text{s.t. } A_i \bullet xx^\top \geq b_i \quad$ for $i = 1, \ldots, m$.

Now, using the spectral theorem for symmetric matrices, one can verify that $X = xx^\top \iff X \succeq 0, \ \text{rank}(X) \leq 1$.

Reason: (1) $\forall u \in \mathcal{U} : u^\top Xu = u^\top xx^\top u = \|x^\top u\|_2^2 \geq 0$.

(2) Every row of X is multiple of every other row.

It follows that problem (9) is equivalent to the following rank-constrained SDP:

$\min\ C \bullet X$

s.t. $A_i \bullet X \geq b_i$ for $i = 1, \ldots, m, \ X \succeq 0, \ \text{rank}(X) \leq 1. \quad (10)$

It reveals where the difficulty of the problem lies; namely, in the non-convex constraint $\text{rank}(X) \leq 1$. By dropping this constraint, we obtain the following semidefinite relaxation of problem (9):

$\min\ C \bullet X \quad \text{s.t. } A_i \bullet X \geq b_i$ for $i = 1, \ldots, m, \ X \succeq 0. \quad (11)$

Problem (11) is an SDP and can be efficiently solved. However, an optimal solution X^* to problem (11) may not be feasible for problem (9).

5.3.4 An Approximation Algorithm for Maximum Cut in Graphs

Suppose that we are given a simple undirected graph $G = (V, E)$ and a function $w : E \to \mathbb{R}_+$ that assigns to each edge $e \in E$ a non-negative weight w_e. The Maximum Cut Problem (MAX-CUT) is that of finding a set $S \subseteq V$ of vertices such that the total weight of the edges in the cut $(S, V \setminus S)$; i.e., sum of the weights of the edges with one endpoint in S and the other in $V \setminus S$, is maximized. By setting $w_{ij} = 0$ if $(i, j) \notin E$, we may denote the weight of a cut $(S, V \setminus S)$ by $w(S, V \setminus S) = \sum_{i\in S, j\in V\setminus S} w_{ij}, \quad (15)$ and our goal is to choose a set $S \subseteq V$ such that the quantity in (15) is maximized.

Let (G, w) be a given instance of the MAX-CUT problem, with $n = |V|$. Then, we can formulate the problem as an integer quadratic program: $v^* = \max\ \frac{1}{2}\sum_{(i,j)\in E} w_{ij}(1 - x_i x_j) \quad \text{s.t. } x_i^2 = 1$ for $i = 1, \ldots, n. \quad (16)$

Here, the variable x_i indicates which side of the cut vertex i belongs to. Note that if vertices i and j belong to the same side of a cut, then $x_i = x_j$, and hence its contribution to the objective function in (16) is zero. Otherwise, its contribution to the objective function is $w_{ij}(1 - (-1))/2 = w_{ij}$.

If let $X = xx^\top \in \mathbb{R}^{n\times n}$, then problem (16) arrive at following relaxation:

$v^*_{sdp} = \max\ \frac{1}{2}\sum_{(i,j)\in E} w_{ij}(1 - X_{ij})$

s.t. $\text{diag}(X) = e, \ X \succeq 0. \quad (18)$. Note that (18) is an SDP. $v^*_{sdp} \geq v^*$.

5.4 Example Problems

1. Let $A \in S^n$ be given. Consider the following QCQP:
   $\min\ x^\top Ax \quad \text{s.t. } x_i^2 = 1$ for $i = 1, \ldots, n$.
   (a) Derive the semidefinite relaxation of Problem.
   (b) Write down the dual of the semidefinite relaxation. Does the primal-dual pair of SDPs you obtained have zero duality gap?
   (c) The Lagrangian dual is $\sup_{w\in\mathbb{R}^n} \theta(w)$, where
   $\theta(w) = \inf_{x\in\mathbb{R}^n}\left\{x^\top Ax + \sum_{i=1}^n w_i(1 - x_i^2)\right\}$.
   Find expression for $\theta(w)$. Hence, or otherwise, show that Lagrangian dual is equivalent to the dual of the semidefinite relaxation found in (b).
   A: (a) The semidefinite relaxation of the given QCQP is given by:
   $\inf\ A \bullet X \quad \text{s.t. } X_{ii} = 1$ for $i = 1, \ldots, n, \ X \succeq 0$.
   (b) The dual of (SDR) is given by: $\quad \sup\ e^\top y \quad \text{s.t. } A - \text{Diag}(y) \succeq 0$.
   Note that $\bar{X} = I$ is strictly feasible for (SDR). It follows from the CLP strong duality theorem that the duality gap between (SDR) and (SDD) is zero.
   (c) $\theta(w) = e^\top w + \inf_{x\in\mathbb{R}^n}\left\{x^\top(A - \text{Diag}(w))x\right\}$.
   For any given $w \in \mathbb{R}^n$, we claim that
   $\inf_{x\in\mathbb{R}^n}\left\{x^\top(A - \text{Diag}(w))x\right\} = \begin{cases} 0 & \text{if } A - \text{Diag}(w) \succeq 0, \\ -\infty & \text{otherwise.} \end{cases}$
   Indeed, if $A - \text{Diag}(w) \not\succeq 0$, then $\lambda_{\min}(A - \text{Diag}(w)) < 0$. Let $u \in \mathbb{R}^n$ be the unit eigenvector corresponding to the smallest eigenvalue of $A - \text{Diag}(w)$. Then, as $\alpha \uparrow +\infty$, we have $(\alpha u)^\top(A - \text{Diag}(w))(\alpha u) = \alpha^2\lambda_{\min}(A - \text{Diag}(w)) \to -\infty$, implies $\inf_{x\in\mathbb{R}^n}\left\{x^\top(A - \text{Diag}(w))x\right\} = -\infty$.
   On the other hand, if $A - \text{Diag}(w) \succeq 0$, then $x^\top(A - \text{Diag}(w))x \geq 0$ for any $x \in \mathbb{R}^n$. In particular, we have $\inf_{x\in\mathbb{R}^n}\left\{x^\top(A - \text{Diag}(w))x\right\} = 0$.
   Consequently, the Lagrangian dual is $\Leftrightarrow$ $\sup\ e^\top w \quad \text{s.t. } A - \text{Diag}(w) \succeq 0$.
2. Consider the following SDP: $\inf\ X_{11} \quad \text{s.t. } \begin{bmatrix} X_{11} & 1 \\ 1 & X_{22} \end{bmatrix} \succeq 0$.
   A: (a) $\inf\ C \bullet X \quad \text{s.t. } \ A \bullet X = 2, \ X \succeq 0$, where
   $C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{bmatrix}$.
   Hence, the dual (D) is given by: $\sup\ 2y \quad \text{s.t. } S = \begin{bmatrix} 1 & -y \\ -y & 0 \end{bmatrix} \succeq 0$.
   (b) A necessary condition for $S \succeq 0$ is $\det(S) \geq 0$; i.e. $y^2 \leq 0$. Thus, $y = 0$ is the only feasible solution to (D), which implies the optimal value of (D) (and hence of (P)) is 0. The dual optimal value is attained by $y = 0$. On the other hand, the primal optimal value is not attained. Indeed, the feasible set of (P) is given by $\{X \in S^2 : X_{11} \geq 0, \ X_{22} \geq 0, \ X_{11}X_{22} \geq 1\}$, which implies $X(\epsilon) = \begin{bmatrix} \epsilon & 1 \\ 1 & \epsilon^{-1} \end{bmatrix}$ is feasible for (P) for any $\epsilon > 0$. However, any point $X \in S^2$ with $X_{11} = 0$ is not feasible for (P).
   21Q5.
   $$\min_{x\in\mathbb{R}^n}\ c^\top x \quad \text{s.t. } d_i^\top x \leq \ell_i, \ i = 1, \ldots, m, \qquad f_j^\top x = g_j, \ j = 1, \ldots, p,$$
   $$x_i \geq 0, \ i = 1, \ldots, n.$$
   $c \in \mathbb{R}^n$ is a positive vector, $d_i \in \mathbb{R}^n$ and $f_j \in \mathbb{R}^n$ are non-negative vectors.
   (a) Write down the KKT conditions for the above problem.
   (b) Suppose $\sum_{j=1}^p \|f_j\|_0 \leq \bar{p} \leq n$, where $\|f_j\|_0$ counts the number of non-zero elements in the vector f_j. Show that there must exist an optimal solution x^* s.t. $\|x^*\|_0 \leq \bar{p}$.
   A: (a) DF: $c + \sum_{i=1}^m u_i d_i + \sum_{j=1}^p w_j f_j + \sum_{k=1}^n v_k(-e_k) = 0$;
   $u \in \mathbb{R}^m_+, \ v \in \mathbb{R}^n_+$.
   CS: $v_i(d_i^\top\bar{x} - \ell_i) = 0 \, \forall i = 1, \ldots, m; \quad u_k\bar{x}_k = 0 \, \forall k = 1, \ldots, n$.
   (b) KKT necessary by (3), sufficient.
   By the KKT condition, we know $\sum_{j=1}^p w_j f_j = u - \sum_{i=1}^m v_i d_i - c$.
   Then $\left\|u - \sum_{i=1}^m v_i d_i - c\right\|_0 \leq \bar{p} \leq n$. Since $u \in \mathbb{R}^n_+$, $d_i \geq 0, \forall i, c > 0$, then as $c + \sum_{i=1}^m v_i d_i$ is positive, then $\left\|u - \sum_{i=1}^m v_i d_i - c\right\|_0 \geq n - \|u\|_0$(this is because at least the zeros of u will become non-zero). Thus $\|u\|_0 \geq n - \bar{p}$. Then by the KKT condition that $u_k x_k^* = 0, k = 1, \ldots, n$, we know that $\|x^*\|_0 \leq \bar{p}$.
3. **19Q4.** Show (P) has a unique optimal solution x^* that is non-degenerate, then (D) also has a unique optimal solution y^* that is non-degenerate.
   A: non-degenerate $\Rightarrow$ $\mathbf{x}^*$ has positive feasible variables (exactly $n - m$ zeros).
   WLOG, assume $\mathbf{x}^* = (x_1, \ldots, x_m, x_{m+1}, \ldots, x_n)$. It follows that rows of A are independent by definition of non-degenerate. By Complementary Slackness Theorem,
   $\left[C_i - (\mathbf{A}^\top\mathbf{y})_i\right] = 0$ for $i = 1, \ldots, m; \quad \left[C_j - (\mathbf{A}^\top\mathbf{y})_j\right] \geq 0$ for $j = m + 1, \ldots, n$.
   Let $\mathbf{A} = \begin{bmatrix}\mathbf{A}_B & \mathbf{A}_N\end{bmatrix}$, $\mathbf{A}^\top\mathbf{y} = \begin{bmatrix}\mathbf{A}_B^\top\mathbf{y} \\ \mathbf{A}_N^\top\mathbf{y}\end{bmatrix} \begin{matrix} = \mathbf{c}_B \\ \leq \mathbf{c}_N \end{matrix}$, Since $\mathbf{A}_B$ is invertible, $\mathbf{y}$ is uniquely determined by $(\mathbf{A}_B^\top)^{-1}\mathbf{c}_B$. $\mathbf{y}$ has exactly m linearly independent active constraints (from
4. $X = \{(t, x_1, x_2) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R} : x_1, x_2 \geq 0, t \leq \sqrt{x_1x_2}\}$ SOC-representable.
   A: Although t is not necessarily non-negative, observe that $t \leq \sqrt{x_1x_2} \ \Leftrightarrow \ t \leq \tau, \ 0 \leq t \leq \sqrt{x_1x_2} \Leftrightarrow \ t \leq \tau, \ \tau \geq 0, \ \tau^2 \leq \frac{(x_1+x_2)^2}{4} - \frac{(x_1-x_2)^2}{4}$
   $\Leftrightarrow \ t \leq \tau, \ \tau \geq 0, \ \left\|\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} \tau \\ x_1 \\ x_2 \end{pmatrix}\right\|_2 \leq \frac{x_1+x_2}{2} \Leftrightarrow$
5. The goal is to prove a theorem of alternatives for linear matrix inequality systems. Let $A_1, \ldots, A_m \in S^n$ and $b \in \mathbb{R}^m$ be given. Suppose $C = \{(A_1 \bullet X, \ldots, A_m \bullet X) : X \succeq 0\} \subseteq \mathbb{R}^m$ is closed. Show that exactly one of the following systems has a solution:
   $$\text{(I)} \quad \begin{cases} A_i \bullet X = b_i & \text{for } i = 1, \ldots, m, \\ X \succeq 0. \end{cases}$$
   $$\text{(II)} \quad \begin{cases} \sum_{i=1}^m y_i A_i \succeq 0, \\ b^\top y = -1. \end{cases}$$
   A: We first show systems (I) and (II) cannot simultaneously have solutions. Suppose not the case. Then, exist a matrix $\bar{X} \in S^n$ and a vector $\bar{y} \in \mathbb{R}^m$ satisfying (I) and (II), respectively. This implies that
   $0 \leq \left(\sum_{i=1}^m \bar{y}_i A_i\right) \bullet \bar{X} \quad$ (since $\bar{X} \succeq 0$ and $\sum_{i=1}^m \bar{y}_i A_i \succeq 0$)
   $= \sum_{i=1}^m \bar{y}_i(A_i \bullet \bar{X}) = \sum_{i=1}^m \bar{y}_i b_i = -1 \quad$ (since $b^\top\bar{y} = -1$) contradiction.
   Now, suppose that system (I) does not have a solution. Then, we have $b \notin C$. Clearly, the set C is non-empty and convex, and by assumption it is closed as well. Hence, by Separation theorem, there exists a vector $s \in \mathbb{R}^m$ such that $\sup_{z\in C} s^\top z < s^\top b$.
   We claim $\sup_{z\in C} s^\top z = 0$. Indeed, since $0 \in C$, we have $\sup_{z\in C} s^\top z \geq 0$. Suppose $\sup_{z\in C} s^\top z > 0$. Then, there exists a matrix $X' \succeq 0$ s.t. $\sum_{i=1}^m s_i(A_i \bullet X') > 0$. In particular, for any $\alpha > 0$, we have $\alpha X' \succeq 0$ and $0 < \alpha\sum_{i=1}^m s_i(A_i \bullet X') = \sum_{i=1}^m s_i(A_i \bullet (\alpha X')) \leq \sup_{z\in C} s^\top z < s^\top b$. However, since $s^\top b$ is a constant, the above inequality cannot hold for all values of α. This contradiction shows that $\sup_{z\in C} s^\top z \leq 0$, and hence the claim is established. As a corollary of the claim, we have $s^\top b > 0$. Thus, the vector $\bar{y} = -s/s^\top b \in \mathbb{R}^m$ is well defined. It is immediate that $b^\top\bar{y} = -1$.
   Moreover, the claim implies $\left(\sum_{i=1}^m \bar{y}_i A_i\right) \bullet X \geq 0$ for all $X \succeq 0$, which, by self-duality of S^n_+, is equivalent to $\sum_{i=1}^m \bar{y}_i A_i \succeq 0$. This shows $\bar{y}$ is a solution to (II).

6 Optimality Conditions and Lagrangian Duality

6.1 Introduction

Consider a univariate, twice continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$.

If $\bar{x} \in \mathbb{R}$ is a local minimum of f, then we must have $\left.\frac{df(x)}{dx}\right|_{x=\bar{x}} = 0$, this is a ***necessary*** condition for $\bar{x}$ to be a local minimum. In addition, $\left.\frac{d^2f(x)}{dx^2}\right|_{x=\bar{x}} > 0$ is a ***sufficient*** condition for $\bar{x}$ to be a local minimum.

Theorem 1 (Taylor's Theorem) Let $a, b \in \mathbb{R}$ be such that $a < b$ and let $n \geq 1$ be an integer. Suppose that the function $f : [a, b] \to \mathbb{R}$ satisfies:

1. $f^{(n-1)}$ is continuous on $[a, b]$,
2. $f^{(n)}(t)$ exists for every $t \in (a, b)$.

Let $a \leq t_1 < t_2 \leq b$, and define $P(t) = \sum_{j=0}^{n-1} \frac{f^{(j)}(t_1)}{j!}(t - t_1)^j$.

Then, $\exists t_0 \in [t_1, t_2]$ such that $f(t_2) = P(t_2) + \frac{f^{(n)}(t_0)}{n!}(t_2 - t_1)^n$.

6.2 Unconstrained Optimization Problems

Proposition 1 Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable at $\bar{x} \in \mathbb{R}^n$. If there exists a $d \in \mathbb{R}^n$ such that $\nabla f(\bar{x})^\top d < 0$, then there exists an $\alpha_0 > 0$ such that $f(\bar{x} + \alpha d) < f(\bar{x})$ for all $\alpha \in (0, \alpha_0)$. In other words, d is a descent direction of f at $\bar{x}$.

Proof Since ∇f is continuous at $\bar{x} \in \mathbb{R}^n$ and $\nabla f(\bar{x})^\top d < 0$, there exists an $\alpha_0 > 0$ such that $\nabla f(\bar{x} + \alpha d)^\top d < 0$ for all $\alpha \in [0, \alpha_0)$. Now, consider the function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ defined by $\tilde{f}(\alpha) = f(\bar{x} + \alpha d)$. By the Chain Rule, we have $\frac{d\tilde{f}(\alpha)}{d\alpha} = \nabla f(\bar{x} + \alpha d)^\top d$. Thus, by Theorem 1, for any $\alpha \in (0, \alpha_0)$, there exists a $t_0 \in [0, \alpha_0)$ such that $f(\bar{x} + \alpha d) = \tilde{f}(\alpha) = \tilde{f}(0) + \alpha\nabla f(\bar{x} + t_0 d)^\top d < \tilde{f}(0) = f(\bar{x})$, as desired. □

Corollary 1 (First Order *Necessary* Condition for Unconstrained Optimization) Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable at $\bar{x} \in \mathbb{R}^n$. If $\bar{x}$ is a local minimum, then we have $\nabla f(\bar{x}) = 0$. In particular, we have $\{d \in \mathbb{R}^n : \nabla f(\bar{x})^\top d < 0\} = \emptyset$.

Proof Suppose to the contrary that $\nabla f(\bar{x}) \neq 0$. Let $d = -\nabla f(\bar{x})$. Then, we have $\nabla f(\bar{x})^\top d = -\|\nabla f(\bar{x})\|_2^2 < 0$. Hence, by Proposition 1, there exists an $\alpha_0 > 0$ such that $f(\bar{x} + \alpha d) < f(\bar{x})$ for all $\alpha \in (0, \alpha_0)$, which contradicts the fact that $\bar{x}$ is a local minimum. Thus, we have $\nabla f(\bar{x}) = 0$. □

Proposition 2 (*Necessary* and *Sufficient* for Convex Functions) Let $S \subseteq \mathbb{R}^n$ be an open convex set. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is convex on S and continuously differentiable at $\bar{x} \in S$. Then, $\bar{x}$ is a global minimum in S ***iff*** $\nabla f(\bar{x}) = \mathbf{0}$.

Proof ($\Rightarrow$) by Corollary 1; ($\Leftarrow$) if $\nabla f(\bar{x}) = \mathbf{0}$, then we have $\nabla f(\bar{x})^\top(x - \bar{x}) = 0$ for all $x \in S$. By Handout 2 Theorem 9, $f(x) \geq f(\bar{x}) \, \forall x \in S$. □

Proposition 3 Let $f : \mathbb{R}^n \to \mathbb{R}$ be arbitrary. Then, $\bar{x}$ is a global minimum iff $0 \in \partial f(\bar{x})$.

Proof $\partial f(\bar{x}) = \{s \in \mathbb{R}^n : f(x) \geq f(\bar{x}) + s^\top(x - \bar{x}) \forall x \in \mathbb{R}^n\}$, $\bar{x}$ is a global minimum iff $f(x) \geq f(\bar{x}) = f(\bar{x}) + \mathbf{0}^\top(x - \bar{x}) \forall x \in \mathbb{R}^n$. □

Note: even if f is differentiable at $\bar{x}$, we may not have $\nabla f(\bar{x}) \in \partial f(\bar{x})$ if f is not convex at $\bar{x}$.

Proposition 4 (Second Order *Sufficient* Condition for Unconstrained Optimization) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable at $\bar{x} \in \mathbb{R}^n$. If $\nabla f(\bar{x}) = 0$ and $\nabla^2 f(\bar{x})$ is pd, then $\bar{x}$ is a local minimum.

Proof For any $d \in \mathbb{R}^n$ such that $\|d\|_2^2 = 1$, consider the function $\tilde{f}_d : \mathbb{R} \to \mathbb{R}$ given by $\tilde{f}_d(\alpha) = f(\bar{x} + \alpha d)$. By the Chain Rule, we have

$\frac{d\tilde{f}_d(\alpha)}{d\alpha} = \nabla f(\bar{x} + \alpha d)^\top d, \quad \frac{d^2\tilde{f}_d(\alpha)}{d\alpha^2} = d^\top\nabla^2 f(\bar{x} + \alpha d)d. \quad (4)$

Since $\nabla^2 f$ is continuous at $\bar{x} \in \mathbb{R}^n$ and $\nabla^2 f(\bar{x}) \succ 0$, there exists an $\alpha_0 > 0$ such that for all unit vectors $d \in \mathbb{R}^n$ and for all $\alpha \in [0, \alpha_0)$, we have $\nabla^2 f(\bar{x} + \alpha d) \succ 0$. Now, suppose that $\bar{x}$ is not a local minimum. Then, there exists an $\bar{x}' \in \mathbb{R}^n$ such that $\|\bar{x}' - \bar{x}\|_2 < \alpha_0$ and $f(\bar{x}') < f(\bar{x})$. Let $d = (\bar{x}' - \bar{x})/\|\bar{x}' - \bar{x}\|_2$ and $\alpha = \|\bar{x}' - \bar{x}\|_2$. Then, by (4), Theorem 1, and the fact that $\nabla f(\bar{x}) = 0$, we have $f(\bar{x}) > f(\bar{x}') = f(\bar{x} + \alpha d) = \tilde{f}_d(\alpha) = \tilde{f}_d(0) + \frac{\alpha^2}{2}d^\top\nabla^2 f(\bar{x} + t_0 d)d > \tilde{f}_d(0) = f(\bar{x})$ for some $t_0 \in (0, \alpha_0)$, which is a contradiction. This completes the proof. □

6.3 Constrained Optimization Problems

Let $f, g_1, \ldots, g_{m_1}, h_1, \ldots, h_{m_2} : \mathbb{R}^n \to \mathbb{R}$ be functions that are continuously differentiable on the non-empty open subset X of $\mathbb{R}^n$.

$$\begin{aligned} \inf f(x) \quad \text{s.t.} \quad & g_i(x) \le 0 \quad \text{for } i = 1, \ldots, m_1, \\ & h_j(x) = 0 \quad \text{for } j = 1, \ldots, m_2, \\ & x \in X. \end{aligned} \tag{5}$$

Let $S = \{x \in X : g_i(x) \le 0 \text{ for } i = 1, \ldots, m_1; h_j(x) = 0 \text{ for } j = 1, \ldots, m_2\}$ be the feasible region of (5).

Theorem 2 (The Fritz John *Necessary* Conditions) Let $\bar{x} \in S$ be a local minimum of problem (5). Then, there exist $u \in \mathbb{R}, v_1, \ldots, v_{m_1} \in \mathbb{R}$, and $w_1, \ldots, w_{m_2} \in \mathbb{R}$ such that

$$u\nabla f(\bar{x}) + \sum_{i=1}^{m_1} v_i \nabla g_i(\bar{x}) + \sum_{j=1}^{m_2} w_j \nabla h_j(\bar{x}) = 0,$$

$$u, v_i \ge 0 \quad \text{for } i = 1, \ldots, m_1,$$

$$(u, v_1, \ldots, v_{m_1}, w_1, \ldots, w_{m_2}) \ne 0. \tag{6}$$

Furthermore, in every neighborhood $\mathcal{N}$ of $\bar{x}$, there exists an $x' \in \mathcal{N}$ such that $v_i g_i(x') > 0$ for all $i \in \{1, \ldots, m_1\}$ with $v_i \ne 0$, and $w_j h_j(x') > 0$ for all $j \in \{1, \ldots, m_2\}$ with $w_j \ne 0$. i.e., $v_i g_i(\bar{x}) = 0 \; \forall i = 1, \ldots, m_1$

Remarks:

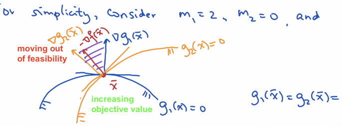
(a) The last statement in Theorem 2 actually implies the complementary slackness condition (i.e., $v_i g_i(\bar{x}) = 0$ for $i = 1, \ldots, m_1$), since if $v_i > 0$, then the corresponding constraint $g_i(x) \le 0$ will be violated by points arbitrarily close to $\bar{x}$. This implies that $g_i(\bar{x}) = 0$.

(b) In Theorem 2, the scalar v_i (resp. w_j) is usually called the **Lagrange multiplier** of the corresponding constraint $g_i(x) \le 0$, where $i = 1, \ldots, m_1$ (resp. $h_j(x) = 0$, where $j = 1, \ldots, m_2$). In a fashion reminiscent to the case of LP, we may summarize the Fritz John necessary conditions in (6) as follows: (7)

$$\begin{aligned} g_i(\bar{x}) \le 0 \quad & \text{for } i = 1, \ldots, m_1, && \text{(primal feasibility I)} \\ h_j(\bar{x}) = 0 \quad & \text{for } j = 1, \ldots, m_2, && \text{(primal feasibility II)} \\ u\nabla f(\bar{x}) + \sum_{i=1}^{m_1} v_i \nabla g_i(\bar{x}) + \sum_{j=1}^{m_2} w_j \nabla h_j(\bar{x}) &= \mathbf{0}, && \text{(dual feasibility I)} \\ u, v_i \ge 0 \quad & \text{for } i = 1, \ldots, m_1, && \text{(dual feasibility II)} \\ (u, v_1, \ldots, v_{m_1}, w_1, \ldots, w_{m_2}) &\ne \mathbf{0}, && \text{(dual feasibility III)} \\ v_i g_i(\bar{x}) = 0 \quad & \text{for } i = 1, \ldots, m_1. && \text{(complementary slackness)} \end{aligned}$$

For any $\bar{x} \in \mathbb{R}^n$, if there exist Lagrange multipliers $u, \{v_i\}_{i=1}^{m_1}, \{w_j\}_{j=1}^{m_2}$ that solve system (7), then we say that $\bar{x}$ is a **Fritz John (FJ) point**. An FJ point need not be a local minimum, as the Fritz John conditions (7) are only necessary conditions for local optimality.

For simplicity, consider $m_1 = 2$, $m_2 = 0$, and



Assuming $u = 1$, $-\nabla f(\bar{x}) = v_1 \nabla g_1(\bar{x}) + v_2 \nabla g_2(\bar{x}); v_1, v_2 \ge 0$.

Idea: If $\bar{x}$ is a local minimum, then intuitively $\nexists d$ s.t. d is simultaneously a descent direction of f at $\bar{x}$ and a feasible direction of f at $\bar{x}$.

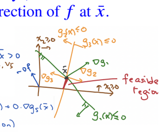
Example (Limitation of FJNC)

At the point $\bar{x}$, $g_1(\bar{x}) = g_2(\bar{x}) = g_3(\bar{x}) = 0$ and $\bar{x} > 0$ (v_1, v_2, v_3, v_4, v_5)

if we let $u = 0$, $v_1 = 0$, $v_2 = v_3 = 1$, $v_4 = v_5 = 0$,

$0 \cdot \nabla f(\bar{x}) + 0 \cdot \nabla g_1(\bar{x}) + 1 \cdot \nabla g_2(\bar{x}) + 1 \cdot \nabla g_3(\bar{x}) + 0 \cdot \nabla g_4(\bar{x}) + 0 \cdot \nabla g_5(\bar{x})$
$= 0$ (because $\nabla g_2(\bar{x}) = -\nabla g_3(\bar{x})$ by construction)

and complementarity holds.



Note: $\bar{x}$ always satisfying FJNC, although maybe not the local minimum. This is because ∇g_2 and ∇g_3 are linearly dependent.

Theorem 3 (The Karush–Kuhn–Tucker *Necessary* Conditions) Let $\bar{x} \in S$ be a local minimum of problem (5). Let $I = \{i \in \{1, \ldots, m_1\} : g_i(\bar{x}) = 0\}$ be the index set for the active constraints. Suppose that $\bar{x}$ is regular; i.e., the family $\{\nabla g_i(\bar{x})\}_{i \in I} \cup \{\nabla h_j(\bar{x})\}_{j=1}^{m_2}$ of vectors is linearly independent (LICQ). Then, there exist $v_1, \ldots, v_{m_1} \in \mathbb{R}$ and $w_1, \ldots, w_{m_2} \in \mathbb{R}$ such that

$$\nabla f(\bar{x}) + \sum_{i=1}^{m_1} v_i \nabla g_i(\bar{x}) + \sum_{j=1}^{m_2} w_j \nabla h_j(\bar{x}) = 0, \text{ (DF)}$$

$$v_i \ge 0 \quad \text{for } i = 1, \ldots, m_1. \text{ (DF)} \tag{17}$$

Furthermore, in every neighborhood $\mathcal{N}$ of $\bar{x}$, $\exists x' \in \mathcal{N}$ such that $v_i g_i(x') > 0$ for all $i \in \{1, \ldots, m_1\}$ with $v_i \ne 0$, and $w_j h_j(x') > 0$ for all $j \in \{1, \ldots, m_2\}$ with $w_j \ne 0$.i.e., $v_i g_i(\bar{x}) = 0 \; \forall i = 1, \ldots, m_1$ (CS)

We say that $\bar{x} \in \mathbb{R}^n$ is a **KKT point** if (i) $\bar{x} \in S$ and (ii) there exist Lagrange multipliers $\{v_i\}_{i=1}^{m_1}, \{w_j\}_{j=1}^{m_2}$ that solve system (17).

Example 1 (Failure of the KKT Conditions in the Absence of Regularity, Importance of CQ) Consider the following problem:

$\min \; x_1 \quad \text{s.t.} \; (x_1 - 1)^2 + (x_2 - 1)^2 \le 1, \; (x_1 - 1)^2 + (x_2 + 1)^2 \le 1.$

Since there is only one feasible solution (i.e., $(x_1, x_2) = (1, 0)$), it is automatically optimal. The KKT conditions are given by

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2v_1 \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} + 2v_2 \begin{bmatrix} x_1 - 1 \\ x_2 + 1 \end{bmatrix} = 0;$ (DF)

$v_1 \left((x_1 - 1)^2 + (x_2 - 1)^2 - 1 \right) = 0, \; v_2 \left((x_1 - 1)^2 + (x_2 + 1)^2 - 1 \right) = 0;$ (CS)

$v_1, v_2 \ge 0.$ (DF)

However, there is no solution $(v_1, v_2) \ge 0$ when $(x_1, x_2) = (1, 0)$.

Without satisfying the CQ, KKT condition is not necessary for optimal point.

Final: If LICQ fails, there may still exist $\bar{v} \in \mathbb{R}^m$ satisfying the KKT condition. For example, if change the above objective function as $\min \; x_2$, then $\bar{x} = (1, 0)$ and $\bar{v} = (1/2, 0)$ is a solution to the KKT conditions.

There are other regularity conditions, a more well-known one is the following:

Theorem 4 Consider problem (5), where $g_1, \ldots, g_{m_1}$ are convex and $h_1, \ldots, h_{m_2}$ are affine. Let $\bar{x} \in S$ be a local minimum and $I = \{i \in \{1, \ldots, m_1\} : g_i(\bar{x}) = 0\}$. Suppose that the Slater condition is satisfied; i.e., there exists an $x' \in S$ such that $g_i(x') < 0$ for $i \in I$ (Slater CQ). Then, $\bar{x}$ satisfies the KKT conditions (17).

Proof Since $h_1, \ldots, h_{m_2}$ are affine, we may assume without loss that the family $\{\nabla h_j(\bar{x})\}_j$ of vectors is linearly independent. Now, by Theorem 2, we have $u\nabla f(\bar{x}) + \sum_{i=1}^{m_1} v_i \nabla g_i(\bar{x}) + \sum_{j=1}^{m_2} w_j \nabla h_j(\bar{x}) = \mathbf{0}$ (19) for some $u, v_1, \ldots, v_{m_1} \ge 0$ and $w_1, \ldots, w_{m_2} \in \mathbb{R}$, where not all of them are zero. We claim that $u > 0$. Suppose that this is not the case. Then, we have

$\sum_{i=1}^{m_1} v_i \nabla g_i(\bar{x}) + \sum_{j=1}^{m_2} w_j \nabla h_j(\bar{x}) = \mathbf{0}.$ (20)

Since not all of $v_1, \ldots, v_{m_1}, w_1, \ldots, w_{m_2}$ are zero, we conclude there exists an $i' \in I$ with $v_{i'} > 0$, for otherwise $\sum_{j=1}^{m_2} w_j \nabla h_j(\bar{x}) = 0$ with some $w_j \ne 0$, which contradicts the linear independence of $\{\nabla h_j(\bar{x})\}_j$.

Now, by the Slater condition and the convexity of $g_1, \ldots, g_{m_1}$, we have

$0 > g_i(x') \ge g_i(\bar{x}) + \nabla g_i(\bar{x})^\top (x' - \bar{x}) = \nabla g_i(\bar{x})^\top (x' - \bar{x})$ for $i \in I$. (21)

Moreover, by the feasibility of x' and the affinity of $h_1, \ldots, h_{m_2}$, we have $0 = \nabla h_j(\bar{x})^\top (x' - \bar{x})$ for $j = 1, \ldots, m_2$. (22) Let $d = x' - \bar{x}$. Since $v_1, \ldots, v_{m_1} \ge 0$, $v_i = 0$ for $i \notin I$, and $v_{i'} > 0$, by (21) and (22), we have $\left(\sum_{i=1}^{m_1} v_i \nabla g_i(\bar{x}) + \sum_{j=1}^{m_2} w_j \nabla h_j(\bar{x}) \right)^\top d = v_{i'} \nabla g_i(\bar{x})^\top d < 0$, which contradicts (20). It follows that $u > 0$ as claimed. Now, upon dividing both sides of (19) by u, the desired result follows. □

Theorem 5 Consider problem (5), where $g_1, \ldots, g_{m_1}$ are concave and $h_1, \ldots, h_{m_2}$ are affine. Let $\bar{x} \in S$ be a local minimum. Then, $\bar{x}$ satisfies the KKT conditions (17).

Proof By Theorem 2, $u\nabla f(\bar{x}) + \sum_{i=1}^{m_1} v_i \nabla g_i(\bar{x}) + \sum_{j=1}^{m_2} w_j \nabla h_j(\bar{x}) = \mathbf{0}$ for some $u, v_1, \ldots, v_{m_1} \ge 0$ and $w_1, \ldots, w_{m_2} \in \mathbb{R}$, where not all of them are zero. We claim $u > 0$. Suppose this is not the case; i.e., $u = 0$. By the concavity of $g_1, \ldots, g_{m_1}$ and affinity of $h_1, \ldots, h_{m_2}$, for any $x \in \mathbb{R}^n$, we have $g_i(x) \le g_i(\bar{x}) + \nabla g_i(\bar{x})^\top (x - \bar{x})$ for $i = 1, \ldots, m_1$,
$h_j(x) = h_j(\bar{x}) + \nabla h_j(\bar{x})^\top (x - \bar{x})$ for $j = 1, \ldots, m_2$.
Since $v_i g_i(\bar{x}) = 0$ for $i = 1, \ldots, m_1$ and $h_j(\bar{x}) = 0$ for $j = 1, \ldots, m_2$,

$\sum_{i=1}^{m_1} v_i g_i(x) + \sum_{j=1}^{m_2} w_j h_j(x)$

$\le \sum_{i=1}^{m_1} v_i g_i(x) + \sum_{j=1}^{m_2} w_j h_j(x)$

$+ \left(\sum_{i=1}^{m_1} v_i \nabla g_i(\bar{x}) + \sum_{j=1}^{m_2} w_j \nabla h_j(\bar{x}) \right)^\top (x - \bar{x}) = 0.$ (24)

Now, since $u = 0$, we either have $v_i > 0$ for some $i = 1, \ldots, m_1$ or $w_j \ne 0$ for some $j = 1, \ldots, m_2$. Thus, by Theorem 2, there exists an $x' \in \mathbb{R}^n$ such that $v_i g_i(x') > 0$ for all i with $v_i > 0$ and $w_j h_j(x') > 0$ for all j with $w_j \ne 0$. However, such an x' satisfies $\sum_{i=1}^{m_1} v_i g_i(x') + \sum_{j=1}^{m_2} w_j h_j(x') > 0$, which contradicts (24). □

In particular, Theorem 5 implies that the KKT conditions (17) are necessary for local optimality in a linearly constrained optimization problem.

Example 2 (Optimality Conditions of Some Optimization Problems)

1. Linear Programming. Consider the standard form LP. Since LP contains only linear constraints, the KKT conditions are necessary for optimality.

$$f(x) = c^\top x, \nabla f(x) = c;$$
$$g_i(x) = -x_i = -e_i^\top x, \nabla g_i(x) = -e_i;$$
$$h_j(x) = b_j - a_j^\top x, \nabla h_j(x) = -a_j.$$

Upon letting $v \in \mathbb{R}^n$ (resp. $w \in \mathbb{R}^m$) be the vector of Lagrange multipliers associated with the inequality constraint (resp. equality constraint), we may write the KKT conditions as follows:

$c + \sum_{i=1}^n v_i(-e_i) + \sum_{j=1}^m w_j(-a_j) = \mathbf{0}, \Rightarrow c - v - A^\top w = 0$ (DF)

$v \ge \mathbf{0},$ (DF)

$v_i x_i = 0 \quad \text{for } i = 1, \ldots, n.$ (CS)

Here, $a_j \in \mathbb{R}^n$ is the j-th row of A, where $j = 1, \ldots, m$. The above can be expressed more compactly as $v = c - A^\top w \ge 0$, $v^\top x = 0$, which correspond to the dual feasibility and complementarity conditions for LP.

2. Smallest Eigenvalue of a Symmetric Matrix Let $A \in \mathcal{S}^n$ be given.

$\min \; x^\top A x \quad \text{s.t.} \; \|x\|_2^2 = 1.$

$f(x) = x^\top A x, \nabla f(x) = 2Ax; \; h(x) = 1 - \|x\|_2^2, \nabla h(x) = -2x.$

Since the feasible set is compact and the objective function is continuous, problem has an optimal solution (Weierstrass). Moreover, since the constraint gradient $\nabla(1 - \|x\|_2^2)$ does not vanish at any feasible solution to (26), the regularity condition in Theorem 3 is satisfied (LICQ). Hence, the KKT conditions are necessary for optimality. Upon letting $w \in \mathbb{R}$ be the Lagrange multiplier associated with the equality constraint, we can write the KKT condition as $2Ax - w(2x) = 0$.

This yields $Ax = wx$, which shows that x has to be an eigenvector of A with eigenvalue w. To determine the optimal value w^* of and optimal solution x^* to problem (26), note that $(x^*)^\top A(x^*) = w^* \|x^*\|_2^2 = w^*$. This implies that the objective value is smallest when w^* is the smallest eigenvalue of A, and the optimal solution x^* is an eigenvector of A corresponding to w^*.

3. Optimization of a Matrix Function Let $A \in \mathcal{S}_{++}^n$ and $b \in \mathbb{R}_{++}$ be given. Consider the following problem:

$\inf \; -\log \det Z \quad \text{s.t.} \; A \bullet Z \le b, \; Z \in \mathcal{S}_{++}^n.$

We claim that problem has an optimal solution: To see this, observe that $Z = \left(\frac{b}{\text{tr}(A)} \right) I$ is feasible. Thus, problem is equivalent to

$\inf_{Z \in \mathcal{F}} -\log \det Z$, where

$\mathcal{F} = \{Z \in \mathcal{S}_+^n : A \bullet Z \le b, -\log \det Z \le -n \log(b/\text{tr}(A))\}$.

Now, for any $Z \in \mathcal{F}$, we have $\lambda_{\min}(A)\text{tr}(Z) \le A \bullet Z \le b$. Reason:

$A \bullet Z = \text{tr}(U\Sigma U^\top Z) = \text{tr}(\Sigma U^\top Z U) = \sum_{i=1}^n \lambda_i(A)(U^\top Z U)_{ii}$

$\ge \sum_{i=1}^n \lambda_{\min}(A)(U^\top Z U)_{ii} = \lambda_{\min}(A) \sum_{i=1}^n (U^\top Z U)_{ii}$

$= \lambda_{\min}(A)\,\text{tr}(U^\top Z U) = \lambda_{\min}(A)\,\text{tr}(Z)$. This implies

$\lambda_i(Z) \le b/\lambda_{\min}(A)$ for $i = 1, \ldots, n$. On the other hand, for $i = 1, \ldots, n$,

$-n \log \left(\frac{b}{\text{tr}(A)} \right) \ge -\log \det Z = -\sum_{i=1}^n \log \lambda_i(Z)$

$\ge -\log \lambda_i(Z) - (n-1) \log \left(\frac{b}{\lambda_{\min}(A)} \right)$, which yields

$\lambda_i(Z) \ge \exp \left(n \log \left(\frac{b}{\text{tr}(A)} \right) - (n-1) \log \left(\frac{b}{\lambda_{\min}(A)} \right) \right) > 0$. In particular, we see that $Z \mapsto -\log \det Z$ is continuous on $\mathcal{F}$ and hence $\mathcal{F}$ is closed. Since optimizing a continuous function over a compact set, it has an optimal solution (Weierstrass). This implies problem has an optimal solution.

Since problem contains only linear constraints, the KKT conditions are necessary for optimality. It is known that

$\nabla f(Z) = \nabla(-\log \det Z) = -Z^{-1}, \quad \nabla g(Z) = \nabla(A \bullet Z - b) = A;$

Upon letting $v \in \mathbb{R}$ be the Lagrange multiplier associated with the inequality:

$-Z^{-1} + vA = 0, \quad v \ge 0, \quad v(A \bullet Z - b) = 0.$

From the first equality, we must have $v > 0$ and $Z = A^{-1}/v$. This, together with the third equality, implies that $b = A \bullet Z = \frac{1}{v}(A \bullet A^{-1}) = \frac{n}{v}$.

Hence, we obtain $v = n/b$. Since the above KKT conditions admit a unique solution, we conclude that $Z^* = bA^{-1}/n$ must be the optimal solution.

In the case where (5) is a convex optimization problem, the KKT conditions are sufficient for optimality as well. To prove this, let us first define the *Lagrangian function* $L : X \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ associated with problem (5) by

$$L(x, v, w) = f(x) + \sum_{i=1}^{m_1} v_i g_i(x) + \sum_{j=1}^{m_2} w_j h_j(x).$$

In the case where (5) is a convex optimization problem, the KKT conditions are ***sufficient*** for optimality as well. To prove this, let us first define the *Lagrangian function* $L : X \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ associated with problem (5) by $L(x, v, w) = f(x) + \sum_{i=1}^{m_1} v_i g_i(x) + \sum_{j=1}^{m_2} w_j h_j(x)$.

Theorem 6 Consider problem (5), where X is open and convex, $f, g_1, \ldots, g_{m_1}$ are convex on X, and $h_1, \ldots, h_{m_2}$ are affine. Suppose that $(\bar{x}, \bar{v}, \bar{w}) \in X \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ is a solution to the KKT conditions

$$g_i(\bar{x}) \le 0 \text{ for } i = 1, \ldots, m_1, \quad \text{(PF)}$$
$$h_j(\bar{x}) = 0 \text{ for } j = 1, \ldots, m_2, \quad \text{(PF)}$$
$$\nabla f(\bar{x}) + \sum_{i=1}^{m_1} \bar{v}_i \nabla g_i(\bar{x}) + \sum_{j=1}^{m_2} \bar{w}_j \nabla h_j(\bar{x}) = 0, \quad \text{(DF)}$$
$$\bar{v} \ge 0, \quad \text{(DF)}$$
$$\bar{v}_i g_i(\bar{x}) = 0 \text{ for } i = 1, \ldots, m_1. \quad \text{(CS)}$$

Then, $\bar{x}$ is an optimal solution to (5).

Proof Since the function $x \mapsto L(x, \bar{v}, \bar{w}) = f(x) + \sum_{i=1}^{m_1} \bar{v}_i g_i(x) + \sum_{j=1}^{m_2} \bar{w}_j h_j(x)$ is convex on X, by condition (c) and Proposition 2, we see that $\bar{x}$ is a global minimum of $x \mapsto L(x, \bar{v}, \bar{w})$ in X. This, together with conditions (b), (d), and (e), implies that

$f(\bar{x}) = f(\bar{x}) + \sum_{i=1}^{m_1} \bar{v}_i g_i(\bar{x}) + \sum_{j=1}^{m_2} \bar{w}_j h_j(\bar{x})$

$= \min_{x \in X} \left\{ f(x) + \sum_{i=1}^{m_1} \bar{v}_i g_i(x) + \sum_{j=1}^{m_2} \bar{w}_j h_j(x) \right\}$

$\le \inf_{x \in X; g_i(x) \le 0, i \in [m_1]; h_j(x) = 0, j \in [m_2]} \left\{ f(x) + \sum_{i=1}^{m_1} v_i g_i(x) + \sum_{j=1}^{m_2} \ldots \right.$

$\le \inf_{x \in X; g_i(x) \le 0, i \in [m_1]; h_j(x) = 0, j \in [m_2]} f(x)$. □

It is important to note that Theorem 6 assumes the existence of the Lagrange multipliers $\bar{v} \in \mathbb{R}^{m_1}$ and $\bar{w} \in \mathbb{R}^{m_2}$. Thus, it does not contradict the observation we made in Example 1.

Conclusion:

Necessary (Theorems 3-5):

(1) LICQ; (2) g convex, h affine, Slater CQ; (3) g concave, h affine

Sufficient (Theorem 6):

X open and convex, f convex, g convex, h affine, optimal exists

Example 3 (Power Allocation Optimization in Parallel AWGN Channels) Consider n parallel additive white Gaussian noise (AWGN) channels. For $i = 1, \ldots, n$, the i-th channel is characterized by the channel power gain $h_i \ge 0$ and the additive Gaussian noise power $\sigma_i > 0$. Let p_i denote the transmit power allocated to the i-th channel, where $i = 1, \ldots, n$. The maximum information rate that can be reliably transmitted over the i-th channel is then given by $r_i = \log_2 \left(1 + \frac{h_i p_i}{\sigma_i} \right) = (\ln 2)^{-1} \ln \left(1 + \frac{h_i p_i}{\sigma_i} \right)$;

Given a budget P on the total transmit power over n channels, our goal is to allocate power $p_1, \ldots, p_n$ on each of the n channels such that the sum rate of all the channels is maximized. We are led to the formulation:

$$\max \sum_{i=1}^n \ln \left(1 + \frac{h_i p_i}{\sigma_i} \right) \quad \text{s.t.} \; \sum_{i=1}^n p_i \le P; \; p_i \ge 0 \text{ for } i = 1, \ldots, n.$$

It is easy to verify that the objective function is concave. Hence, problem is a linearly constrained convex maximization problem. Now, by Theorems 5 and 6, every solution $(\bar{p}, \bar{v}) \in \mathbb{R}^n \times \mathbb{R}^{n+1}$ to the following KKT system will yield an optimal solution $\bar{p} \in \mathbb{R}^n$:

$v_0 - v_i = \frac{h_i}{h_i p_i + \sigma_i} \quad \text{for } i = 1, \ldots, n,$ (DF1)

$v_0 \left(\sum_{i=1}^n p_i - P \right) = 0,$ (CS1)

$v_i p_i = 0 \quad \text{for } i = 1, \ldots, n,$ (CS2)

$v_i \ge 0 \quad \text{for } i = 0, 1, \ldots, n.$ (DF2)

we may assume $h_i > 0$ for $i = 1, \ldots, n$. Then, we have $v_0 \ge v_i \ge 0$ by two (DF)s, which implies that $p_i = \frac{1}{v_0 - v_i} - \frac{\sigma_i}{h_i}$ for $i = 1, \ldots, n$. (32)

Now, if $p_i > 0$, then $v_i = 0$ by (CS2). On the other hand, if $p_i = 0$, then in order to satisfy (32) with some $v_i \ge 0$, we must have $\frac{1}{v_0} - \frac{\sigma_i}{h_i} \le 0$.

Hence, we obtain $p_i = \left(\frac{1}{v_0} - \frac{\sigma_i}{h_i} \right)^+ \quad \text{for } i = 1, \ldots, n.$

Moreover, since $v_0 > 0$, $\sum_{i=1}^n p_i = P$ by (CS1). $\sum_{i=1}^n \left(\frac{1}{v_0} - \frac{\sigma_i}{h_i} \right)^+ = P$.

In particular, we can solve for the unique positive root $\bar{v}_0$ of the above equation by a simple bisection search over the interval $0 < v_0 < \max_i (h_i/\sigma_i)$. Once we have $\bar{v}_0$, we can extract the optimal allocation $\bar{p} = (\bar{p}_1, \ldots, \bar{p}_n)$.

6.4 Lagrangian Duality

$$\begin{aligned} \inf f(x) \quad \text{s.t.} \quad & g_i(x) \le 0 \quad \text{for } i = 1, \ldots, m_1, \\ & h_j(x) = 0 \quad \text{for } j = 1, \ldots, m_2, \\ & x \in X. \end{aligned} \tag{P}$$

Here, $f, g_1, \ldots, g_{m_1}, h_1, \ldots, h_{m_2} : \mathbb{R}^n \to \mathbb{R}$ are arbitrary functions, and X is an arbitrary non-empty subset of $\mathbb{R}^n$.

We write the first two sets of constraints in (P) as $G(x) \le 0$ and $H(x) = 0$, where $G : \mathbb{R}^n \to \mathbb{R}^{m_1}$ is given by $G(x) = (g_1(x), \ldots, g_{m_1}(x))$ and $H : \mathbb{R}^n \to \mathbb{R}^{m_2}$ is given by $H(x) = (h_1(x), \ldots, h_{m_2}(x))$.

Reformulate (P) using a penalty function approach. (P) is equivalent to $\inf_{x \in X} \sup_{v \in \mathbb{R}_+^{m_1}, w \in \mathbb{R}^{m_2}} L(x, v, w)$, where

$L : \mathbb{R}^n \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ is the Lagrangian function associated with (P):

$L(x, v, w) = f(x) + v^\top G(x) + w^\top H(x)$.

This follows from the fact that for any $x \in X$, (35)

$$\sup_{v \in \mathbb{R}_+^{m_1}, w \in \mathbb{R}^{m_2}} \left\{ f(x) + v^\top G(x) + w^\top H(x) \right\} = \begin{cases} f(x) & \text{if } G(x) \le 0 \text{ an…} \\ +\infty & \text{otherwise.} \end{cases}$$

Penalize by letting $v \uparrow +\infty$, letting $w \uparrow +\infty$ or $w \downarrow -\infty$.

Now, it is clear that for any $\bar{x} \in X$ and $(\bar{v}, \bar{w}) \in \mathbb{R}_+^{m_1} \times \mathbb{R}^{m_2}$, $\theta(\bar{v}, \bar{w}) = \inf_{x \in X} L(x, \bar{v}, \bar{w}) \le L(\bar{x}, \bar{v}, \bar{w})$

$\le \sup_{v \in \mathbb{R}_+^{m_1}, w \in \mathbb{R}^{m_2}} L(\bar{x}, v, w) = \gamma(\bar{x})$. (*)

Hence, we have $v_d^* = \sup_{v \in \mathbb{R}_+^{m_1}, w \in \mathbb{R}^{m_2}} \inf_{x \in X} L(x, v, w)$

$\le \inf_{x \in X} \sup_{v \in \mathbb{R}_+^{m_1}, w \in \mathbb{R}^{m_2}} L(x, v, w) = v_p^*$.

Observe that the right-hand side is precisely problem (P). This motivates us to define the following dual of (P):

$v_d^* = \sup_{v \in \mathbb{R}_+^{m_1}, w \in \mathbb{R}^{m_2}} \theta(v, w),$ (D)

where $\theta : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ is the value function given by

$\theta(v, w) = \inf_{x \in X} L(x, v, w)$.

Problem (D) is known as a **Lagrangian dual** of problem (P).

Since the set X is arbitrary, there can be different Lagrangian duals for the same primal problem, depending on which constraints are handled as $G(x) \le 0$ and $H(x) = 0$, and which constraints are treated by X. Different choices of the Lagrangian dual problem may lead to different dual optimal value.

Theorem 7 (Weak Duality Theorem) Let $\bar{x} \in \mathbb{R}^n$ feasible for (P) and $(\bar{v}, \bar{w}) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ feasible for (D). Then, $\theta(\bar{v}, \bar{w}) \le \gamma(\bar{x}) = f(\bar{x})$.

Note that the value function θ is the pointwise infimum of affine functions. As such, it is a *concave* function, regardless of the convexity of (P). In particular, the Lagrangian dual (D) is always a *convex* optimization problem. So strong duality between (P) and (D) may not hold in general.

Example 4 (A Primal-Dual Pair with Non-Zero Duality Gap)
$v_p^* = \min_{x\in X} -x \quad \text{s.t. } x \le 1,\ x \in X = \{0, 2\}$.
It is clear that the optimal value of and optimal solution are $v_p^* = 0$ and $x^* = 0$, respectively. By dualizing the inequality constraint, we obtain the following Lagrangian dual: $v_d^* = \sup_{v\ge 0} \min_{x\in\{0,2\}} \{-x + v(x-1)\}$.
Also we can: $v_d^* = \sup_{v\ge 0} \min_{x\in\mathbb{R}} \{-x + v(x-1) + w(x(x-2))\}$.
Observe that for any $v \ge 0$, we have
$\min_{x\in\{0,2\}} \{-x + v(x-1)\} = \min\{-v, v-2\}$.
It follows that the optimal value of and optimal solution to (38) are $v_d^* = -1$ and $v^* = 1$, respectively. In this case, we have $v_p^* > v_d^*$.

Definition 1 We say that $(\bar{x}, \bar{v}, \bar{w}) \in \mathbb{R}^n \times \mathbb{R}_+^{m_1} \times \mathbb{R}^{m_2}$ is a **saddle point** of the Lagrangian function L of (P) if the following conditions are satisfied:
(a) $\bar{x} \in X$. (b) $\bar{v} \ge 0$. (c) For all $x \in X$ and $(v, w) \in \mathbb{R}_+^{m_1} \times \mathbb{R}^{m_2}$, $L(\bar{x}, v, w) \le L(\bar{x}, \bar{v}, \bar{w}) \le L(x, \bar{v}, \bar{w})$.
In particular, the point $(\bar{x}, \bar{v}, \bar{w})$ is a saddle point of L if $\bar{x}$ minimizes L over all $x \in X$ when (v, w) is fixed at $(\bar{v}, \bar{w})$, and $(\bar{v}, \bar{w})$ maximizes L over all $(v, w) \in \mathbb{R}_+^{m_1} \times \mathbb{R}^{m_2}$ when x is fixed at $\bar{x}$.
Def + (*): $L(\bar{x}, \bar{v}, \bar{w}) = \inf_{x\in X} L(x, \bar{v}, \bar{w}) = \sup_{x\in X} L(\bar{x}, v, w)$.

Theorem 8 The point $(\bar{x}, \bar{v}, \bar{w}) \in \mathbb{R}^n \times \mathbb{R}_+^{m_1} \times \mathbb{R}^{m_2}$ is a saddle point of the Lagrangian function L associated with (P) iff the duality gap between (P) and (D) is zero and $\bar{x}$ and $(\bar{v}, \bar{w})$ are the optimal solutions to (P) and (D).
Proof Suppose that $(\bar{x}, \bar{v}, \bar{w})$ is a saddle point of L. From condition (c), $L(\bar{x}, v, w) \le L(\bar{x}, \bar{v}, \bar{w}) \le L(x, \bar{v}, \bar{w})\ \forall (v, w) \in \mathbb{R}_+^{m_1} \times \mathbb{R}^{m_2}$. It follows from condition (a) and the identity (35) that $\bar{x}$ is feasible for (P). It is also clear from condition (b) that $(\bar{v}, \bar{w})$ is feasible for (D). Hence, by condition (c), we have
$$\theta(\bar{v}, \bar{w}) = \min_{x\in X} L(x, \bar{v}, \bar{w}) = L(\bar{x}, \bar{v}, \bar{w}) = \max_{v\in\mathbb{R}_+^{m_1}, w\in\mathbb{R}^{m_2}} L(\bar{x}, v, w) = f(\bar{x}),$$
i.e., the duality gap between (P) and (D) is zero, and the common optimal value $v_p^* = v_d^*$ is attained by the primal solution $\bar{x}$ and dual solution $(\bar{v}, \bar{w})$.
Conversely, suppose that $\bar{x}$ and $(\bar{v}, \bar{w})$ are optimal for (P) and (D), respectively, with $f(\bar{x}) = \theta(\bar{v}, \bar{w})$. Then, we have $\bar{x} \in X$, $G(\bar{x}) \le 0$, $H(\bar{x}) = 0$, and $\bar{v} \ge 0$; i.e., conditions (a) and (b) are satisfied. Moreover, by the primal feasibility of $\bar{x}$ and dual feasibility of $(\bar{v}, \bar{w})$, we have
$$\theta(\bar{v}, \bar{w}) = \inf_{x\in X} L(x, \bar{v}, \bar{w}) \le L(\bar{x}, \bar{v}, \bar{w}) \le \sup_{v\in\mathbb{R}_+^{m_1}, w\in\mathbb{R}^{m_2}} L(\bar{x}, v, w) = f(\bar{x}).$$
Since we have $f(\bar{x}) = \theta(\bar{v}, \bar{w})$ by assumption, equality must hold throughout the above chain of inequalities. In particular, for any $x \in X$ and $(v, w) \in \mathbb{R}_+^{m_1} \times \mathbb{R}^{m_2}$, we have
$$L(\bar{x}, v, w) \le \sup_{v\in\mathbb{R}_+^{m_1}, w\in\mathbb{R}^{m_2}} L(\bar{x}, v, w) = L(\bar{x}, \bar{v}, \bar{w}) = \inf_{x\in X} L(x, \bar{v}, \bar{w}) \le L(x, \bar{v}, \bar{w});$$
i.e., condition (c) is satisfied. This completes the proof. □
$$\sup_{v\in\mathbb{R}_+^{m_1}, w\in\mathbb{R}^{m_2}} \inf_{x\in X} L(x, v, w) = \inf_{x\in X} \sup_{v\in\mathbb{R}_+^{m_1}, w\in\mathbb{R}^{m_2}} L(x, v, w).$$

Theorem 9 Let L be the Lagrangian function associated with (P). Suppose (a) X is a compact convex subset of $\mathbb{R}^n$; (b) $(v, w) \mapsto L(x, v, w)$ is continuous and concave on $\mathbb{R}_+^{m_1} \times \mathbb{R}^{m_2}$ for each $x \in X$; (c) $x \mapsto L(x, v, w)$ is continuous and convex on X for each $(v, w) \in \mathbb{R}_+^{m_1} \times \mathbb{R}^{m_2}$.
Then, we have (strong duality)
$$\sup_{v\in\mathbb{R}_+^{m_1}, w\in\mathbb{R}^{m_2}} \min_{x\in X} L(x, v, w) = \min_{x\in X} \sup_{v\in\mathbb{R}_+^{m_1}, w\in\mathbb{R}^{m_2}} L(x, v, w).$$

Theorem 10 (Saddle Point Optimality Conditions)
The point $(\bar{x}, \bar{v}, \bar{w}) \in \mathbb{R}^n \times \mathbb{R}_+^{m_1} \times \mathbb{R}^{m_2}$ is a saddle point of the Lagrangian function L associated with (P) iff the following hold:
(a) *(Primal Feasibility)* $\bar{x} \in X$, $G(\bar{x}) \le 0$, and $H(\bar{x}) = 0$.
(b) *(Lagrangian Optimality)* $\bar{v} \ge 0$ and $\bar{x} = \arg\min_{x\in X} L(x, \bar{v}, \bar{w})$.
(c) *(Complementarity)* $\bar{v}^\top G(\bar{x}) = 0$.
Proof Suppose that $(\bar{x}, \bar{v}, \bar{w})$ is a saddle point of L. Then, conditions (a) and (b) follow from Definition 1 and Theorem 8. Now, Definition 1 implies that $f(\bar{x}) = L(\bar{x}, 0, 0) \le L(\bar{x}, \bar{v}, \bar{w}) = f(\bar{x}) + \bar{v}^\top G(\bar{x})$, or equivalently, $\bar{v}^\top G(\bar{x}) \ge 0$. On the other hand, since $\bar{v} \ge 0$ and $G(\bar{x}) \le 0$, we have $\bar{v}^\top G(\bar{x}) \le 0$. This gives condition (c).
Conversely, suppose that $(\bar{x}, \bar{v}, \bar{w}) \in \mathbb{R}^n \times \mathbb{R}_+^{m_1} \times \mathbb{R}^{m_2}$ satisfies conditions (a)-(c) above. Then, we have $L(x, \bar{v}, \bar{w}) \le L(\bar{x}, \bar{v}, \bar{w})$ for all $x \in X$. Moreover, we have $L(\bar{x}, \bar{v}, \bar{w}) = f(\bar{x}) + \bar{v}^\top G(\bar{x}) + \bar{w}^\top H(\bar{x}) \ge f(\bar{x}) + \bar{v}^\top G(\bar{x}) + \bar{w}^\top H(\bar{x}) = L(\bar{x}, \bar{v}, \bar{w})\ \forall (v, w) \in \mathbb{R}_+^{m_1} \times \mathbb{R}^{m_2}$, since $\bar{v}^\top G(\bar{x}) = 0$, $G(\bar{x}) \le 0$, and $H(\bar{x}) = 0$. By Definition 1, we conclude that $(\bar{x}, \bar{v}, \bar{w})$ is a saddle point of L. □

Corollary 2 Consider problem (P), where X is open and convex, $f, g_1, \ldots, g_{m_1}$ are convex and continuously differentiable on X, and $h_1, \ldots, h_{m_2}$ are affine. Suppose that (P) has an optimal solution and satisfies the Slater condition. Then, the dual (D) also has an optimal solution. Moreover, we have $v_p^* = v_d^*$.
Proof Let $\bar{x}$ be an optimal solution to (P). By Theorem 4, there exist $\bar{v} \in \mathbb{R}^{m_1}$ and $\bar{w} \in \mathbb{R}^{m_2}$ such that $(\bar{x}, \bar{v}, \bar{w})$ satisfies the KKT conditions of (P). By Proposition 2, $\nabla f(\bar{x}) + \sum_{i=1}^{m_1} \bar{v}_i \nabla g_i(\bar{x}) + \sum_{j=1}^{m_2} \bar{w}_j \nabla h_j(\bar{x}) = 0$ is equivalent to $\bar{x} = \arg\min_{x\in X} L(x, \bar{v}, \bar{w})$. So $(\bar{x}, \bar{v}, \bar{w})$ is a saddle point of the Lagrangian function L associated with (P). It follows from Theorems 8 and 10 that $(\bar{v}, \bar{w})$ is an optimal solution to (D) and $v_p^* = v_d^*$. □

Corollary 3 Consider problem (P), where X is open and convex, f is convex and continuously differentiable on X, and $g_1, \ldots, g_{m_1}, h_1, \ldots, h_{m_2}$ are affine. Suppose that (P) has an optimal solution. Then, the dual (D) also has an optimal solution. Moreover, we have $v_p^* = v_d^*$.
Proof: same as Corollary 2, except invoke Theorem 5 instead of Theorem 4.

6.4.1 Example Problems

1. Let $Q \in S^n$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$ be given. Consider the following problem: $\inf_x\ f(x) = \frac{1}{2}x^\top Qx + c^\top x \quad \text{s.t.} Ax \le b$
   (a) Let $v \in \mathbb{R}^m$ be the multiplier associated with the constraint $Ax \le b$. Write down the KKT conditions associated with Problem and explain why they are necessary for optimality.
   (b) Let $\bar{x} \in \mathbb{R}^n$ be a KKT point of Problem; i.e., there exists a multiplier $\bar{v} \in \mathbb{R}^m$ such that $(\bar{x}, \bar{v})$ satisfies the KKT conditions found in (a). Let $\mathcal{I}(\bar{x}) = \{i : a_i^\top \bar{x} = b_i\}$, where $a_i^\top$ is the i-th row of A, be the active index set associated with $\bar{x}$. Suppose that whenever $d \in \mathbb{R}^n$ satisfies $a_i^\top d \le 0$ for all $i \in \mathcal{I}(\bar{x})$, we have $d^\top Qd \ge 0$. Show $\bar{x}$ is a local minimum of Problem.
   A: (a) The KKT conditions associated with Problem are given by $Qx + c + \sum_{i=1}^m v_i a_i = 0; \quad v \ge 0; \quad v_i(a_i^\top x - b_i) = 0$ for $i = 1, \ldots, m$.
   Since Problem is a linearly constrained optimization problem, by Theorem 5, the above KKT conditions are necessary for optimality.
   (b) Since f is a quadratic function, by Taylor's theorem, $f(x) - f(\bar{x}) = \nabla f(\bar{x})^\top (x - \bar{x}) + \frac{1}{2}(x - \bar{x})^\top Q(x - \bar{x})$ for any $x \in \mathbb{R}^n$. Now, let x be a feasible solution to Problem that is sufficiently close to $\bar{x}$ and set $d = x - \bar{x}$.
   $f(x) - f(\bar{x}) = \nabla f(\bar{x})^\top d + \frac{1}{2}d^\top Qd = -\sum_{i=1}^m \bar{v}_i a_i^\top d + \frac{1}{2}d^\top Qd$
   $= -\sum_{i\in\mathcal{I}(\bar{x})} \bar{v}_i a_i^\top (x - \bar{x}) + \frac{1}{2}d^\top Qd \ge 0$,
   where the second equality follows from the KKT condition (i), the third equality follows from the KKT condition (iii) (since $\bar{v}_i = 0$ if $i \notin \mathcal{I}(\bar{x})$), and the last inequality follows from the KKT condition (ii) and our assumption on Q (since $a_i^\top d = a_i^\top (x - \bar{x}) = a_i^\top x - b_i \le 0$ for all $i \in \mathcal{I}(\bar{x})$).
2. Consider the problem $\min_{x\in\mathbb{R}^n} \max\{g_1(x), \ldots, g_m(x)\}$. Show that $x^* \in \mathbb{R}^n$ is an optimal solution to Problem (1) if and only if there exists a vector $u^* \in \mathbb{R}^m$ such that
   $\sum_{j=1}^m u_j^* \nabla g_j(x^*) = 0; \quad u^* \ge 0; \quad \sum_{j=1}^m u_j^* = 1;$
   $u_j^* = 0 \quad \text{if} \quad g_j(x^*) < \max\{g_1(x^*), \ldots, g_m(x^*)\}, \quad \text{for} \quad j = 1, \ldots, m.$
   A: Problem equivalent to: $\min z \quad \text{s.t.} g_j(x) \le z$ for $j = 1, \ldots, m$.
   $\begin{bmatrix}0\\1\end{bmatrix} + \sum_{j=1}^m u_j^* \begin{bmatrix}\nabla g_j(x^*)\\-1\end{bmatrix} = 0, \quad u_j^*(g_j(x^*) - z^*) = 0$ for $j = 1, \ldots, m$,
   $g_j(x^*) \le z^* \quad \text{for} \quad j = 1, \ldots, m, \qquad u^* \ge 0.$

7 Simplex Method

Definition Let $S \subseteq \mathbb{R}^n$ be a non-empty convex set, $x \in S$. If for any $\lambda \in (0, 1)$ and $x_1, x_2 \in S$, $x = \lambda x_1 + (1-\lambda)x_2$ implies $x_1 = x_2 = x$, then x is called a **vertex** of S.
Definition Let $S \subseteq \mathbb{R}^n$ be a non-empty convex set, $d \in \mathbb{R}^n$. If for any $x \in S$ and $\lambda \ge 0$ such that $x + \lambda d \in S$, then d is called a **direction** of S. If for any directions d_1, d_2 of S, $d = \alpha d_1 + \lambda d_2$ where $\alpha > 0$, then d is called an **extreme direction** of S.
Let $S = \{x \in \mathbb{R}^n \mid Ax = b, x \ge 0\}$, where $A \in \mathbb{R}^{m\times n}$ is full row rank. By definition, $d \ne 0$ is a direction of S iff $Ad = 0, d \ge 0$.
Let $A = (B, N)$, where $B \in \mathbb{R}^{m\times m}$ is non-singular, $N \in \mathbb{R}^{m\times(n-m)}$. Decompose x as $x^\top = (x_B^\top, x_N^\top)$. Then $x \in S$ can be written as $Bx_B + Nx_N = b,\ x_B \ge 0,\ x_N \ge 0$. Thus, $x_B = B^{-1}(b - Nx_N)$.
Let $x_N = 0$, then $x_B = B^{-1}b$. If $B^{-1}b \ge 0$, then $x = \begin{pmatrix}B^{-1}b\\0\end{pmatrix}$ is a vertex of S. In fact, if there exists $\lambda \in (0, 1)$ and $x_1, x_2 \in S$ such that $x = \lambda x_1 + (1-\lambda)x_2$, then $x_1 = x_2 = x$.
Theorem Let $S = \{x \in \mathbb{R}^n \mid Ax = b, x \ge 0\}$, where A is full row rank. Then $x \in S$ is a vertex of S iff x can be expressed as $x = \begin{pmatrix}B^{-1}b\\0\end{pmatrix}$, where $A = (B, N)$, B is invertible and $B^{-1}b \ge 0$.
Proof Only need to prove necessity. Let $x \in S$ be a vertex of S. W.l.o.g., let $x = (x_1, \ldots, x_k, 0, \ldots, 0)^\top$, where $x_i > 0$, $i = 1, \ldots, k$. Let $A = (a_1, \ldots, a_n)$. We claim $a_1, \ldots, a_k$ are linearly independent: If there exist non-zero $\lambda_1, \ldots, \lambda_k$ such that $\sum_{i=1}^k \lambda_i a_i = 0$. Let $\lambda = (\lambda_1, \ldots, \lambda_k, 0, \ldots, 0)^\top$, let $x_1 = x + \alpha\lambda$, $x_2 = x - \alpha\lambda$, then $x = \frac{1}{2}x_1 + \frac{1}{2}x_2$. Appropriate choice of $\alpha > 0$ can make $x_1, x_2 \ge 0$ and $x_1 \ne x_2$. Note that $Ax_1 = Ax + \alpha\sum_{i=1}^k \lambda_i a_i = b$, so $x_1 \in S$. Similarly, $x_2 \in S$. This contradicts with x being a vertex, so $a_1, \ldots, a_k$ are linearly independent. Because A is full row rank, we can always choose $a_{k+1}, \ldots, a_m$ from $a_{k+1}, \ldots, a_n$ such that $B = (a_1, \ldots, a_m)$ is invertible. Therefore, $x = \begin{pmatrix}B^{-1}b\\0\end{pmatrix}$, where $B^{-1}b = (x_1, \ldots, x_k, 0, \ldots, 0)^\top \ge 0$. □
Theorem 2.4 Let $S = \{x \in \mathbb{R}^n \mid Ax = b, x \ge 0\}$ be non-empty, where A is full row rank. Then S has at least one vertex.
Proof Let $x \in S$, w.l.o.g., let $x = (x_1, \ldots, x_k, 0, \ldots, 0)^\top$, where $x_i > 0$, $i = 1, \ldots, k$. If $a_1, \ldots, a_k$ are linearly independent, then $k \le m$, so by previous Theorem, x is a vertex of S. Otherwise, there exist non-zero $\lambda_1, \ldots, \lambda_k$ such that $\sum_{i=1}^k \lambda_i a_i = 0$. Without loss of generality, assume $\lambda_1 > 0$. Let $\alpha = \min\left\{\frac{x_i}{\lambda_i} \mid \lambda_i > 0, i = 1, \ldots, k\right\}$. Construct $\tilde{x}$ as follows: $\tilde{x}_i = x_i - \alpha\lambda_i$, $i = 1, \ldots, k$; $\tilde{x}_i = 0$, $i = k+1, \ldots, n$. Then $\tilde{x} \in S$ and the number of its non-zero components is at most $k - 1$. This process continues, and we will eventually find $\tilde{x} \in S$ with linearly independent non-zero components corresponding to A, and thus $\tilde{x}$ is a vertex of S. □
Theorem 2.5 Let $S = \{x \in \mathbb{R}^n \mid Ax = b, x \ge 0\}$ be non-empty, where A is full row rank. Then $d \in \mathbb{R}^n$ is an extreme direction of S iff there exists a decomposition $A = (B, N)$ and the j-th column a_j of N such that $B^{-1}a_j \le 0$, making $d = t\begin{pmatrix}-B^{-1}a_j\\e_j\end{pmatrix}$, where $t > 0$ and e_j is the j-th unit vector in $\mathbb{R}^{n-m}$.
Theorem 2.6 Let $S = \{x \in \mathbb{R}^n \mid Ax = b, x \ge 0\}$ be non-empty, where A is full row rank. Let the vertices of S be $x_1, \ldots, x_k$, and the extreme directions be $d_1, \ldots, d_l$. Then $S = \left\{\sum_{i=1}^k \lambda_i x_i + \sum_{j=1}^l \mu_j d_j \mid \sum_{i=1}^k \lambda_i = 1, \lambda_i \ge 0, \mu_j \ge 0, i = 1, \ldots, k, j = 1, \ldots, l\right\}$.
Primal Simplex Algorithm Step 0. Compute initial basic feasible solution $x = \begin{pmatrix}B^{-1}b\\0\end{pmatrix}$.
Step 1. If $r_N = c_N^\top - c_B^\top B^{-1}N \ge 0$, stop; current basic feasible solution is optimal, otherwise, go to Step 2.
Step 2. Choose j such that $c_j - c_B^\top B^{-1}a_j < 0$. If $\tilde{a}_j = B^{-1}a_j \le 0$, stop; the problem is unbounded, otherwise, go to Step 3. (calculate z_j's and $(\sigma_j = c_j - z_j)$'s)
Step 3. Compute λ by (2.11), let $x := x + \lambda d_j$, where $d_j = \begin{pmatrix}-B^{-1}a_j\\e_j\end{pmatrix}$.
Return to Step 1. (calculate (b_i/a_{ij}))'s; calculate $z = \langle c_B, b\rangle$)
Example:
$$\begin{aligned} \max \quad & 50x_1 + 100x_2 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 300 \\ & 2x_1 + x_2 + x_4 = 400 \\ & x_2 + x_5 = 250 \end{aligned}$$

Basis	C_B	x_1	x_2	x_3	x_4	x_5	b	b_i/a_{ij}
x_3	0	1	1	1	0	0	300	300/1
x_4	0	2	1	0	1	0	400	400/1
x_5	0	0	1	0	0	1	250	**250/1**
z_j		0	0	0	0	0		
σ_j		50	**100**	0	0	0		$z = 0$

Basis	C_B	x_1	x_2	x_3	x_4	x_5	b	b_i/a_{ij}
x_3	0	1	0	1	0	−1	50	**50/1**
x_4	0	2	0	0	1	−1	150	150/2
x_2	100	0	1	0	0	1	250	250/0
z_j		0	100	0	0	100		
σ_j		50	0	0	0	−100		$z = 25k$

Basis	C_B	x_1	x_2	x_3	x_4	x_5	b	
x_1	50	1	0	1	0	−1	**50**	
x_4	0	0	0	−2	1	1	50	
x_2	100	0	1	0	0	1	**250**	
z_j		50	100	50	0	50		
σ_j		0	0	−50	0	−50	$z = 2.75k$	

8 Integer Linear Programming

$$\max\{cx + hy : (x, y) \in S\}$$
$$S := \{(x, y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p : Ax + Gy \le b\}.$$

8.1 Examples

Assignment: There are m machines and n tasks. The available working hours for machine i is b_i. The working hours required for machine i to complete task j is a_{ij}, and the cost is c_{ij}. How to optimally assign tasks to machines to minimize the cost? Let $x_{ij} = \begin{cases}1, & \text{if machine } i \text{ processes task } j,\\ 0, & \text{otherwise.}\end{cases}$
$$\min \sum_{i=1}^n \sum_{j=1}^n c_{ij}x_{ij} \quad \text{s.t.} \sum_{j=1}^n a_{ij}x_{ij} \le b_i, \quad i = 1, \ldots, m,$$
$$\sum_{i=1}^m x_{ij} = 1, \quad j = 1, \ldots, n, \qquad x_{ij} \in \{0, 1\}$$
Set Covering: A certain area is divided into m districts, and it is required to establish n emergency service centers. The location of each center is known, and each center requires a setup cost. Each center can serve a certain range of districts. How to choose the centers to cover the entire area and minimize the setup cost? Let $M = \{1, \ldots, m\}$ be the districts and $N = \{1, \ldots, n\}$ be the candidate centers. Let $S_j \subseteq M$ be the set of districts that center j can serve, and c_j be the setup cost of center j. Define the 0-1 matrix $A = (a_{ij})$, where $a_{ij} = 1$ if $i \in S_j$, otherwise $a_{ij} = 0$.
Let $x_j = 1$ if center j is selected, 0 otherwise.
$$\min \sum_{j=1}^n c_jx_j \quad \text{s.t.} \sum_{j=1}^n a_{ij}x_j \ge 1, i = 1, \ldots, m; \quad x \in \{0, 1\}^n$$

8.2 Useful Concepts

8.2.1 Unimodular Matrix

Theorem: If the optimal basis matrix B of a LP problem satisfies $\det(B) = \pm 1$, then the optimal solution is an integer solution.
Definition: Let A be $m \times n$ integer matrix. If the determinant of any square submatrix of A is 0, 1, or −1, then A is called a totally unimodular matrix.
Property: If A is totally unimodular, the elements of A are 0, 1, or −1.
Proof: Any submatrix of A has a determinant of 0, 1, or −1 implies the elements a_{ij} are 0, 1, or −1. □
Theorem: Let A be a totally unimodular matrix and b be an integer vector. Then all vertices of polyhedron $P = \{x \in \mathbb{R}_+^n \mid Ax \le b\}$ are integer points.
Proof: The polyhedron P can be represented as $Ax + Iy = b, x \in \mathbb{R}_+^n, y \in \mathbb{R}_+^m$. Let $(A, I) = (B, N)$, where B is the basis matrix. By Property B^{-1} is integer matrix, thus $x^\top = (B^{-1}b, 0)^\top$ is integer vector. Since vertices of the polyhedron correspond to basic feasible solutions, P has integer vertices.
Corollary: *A totally unimodular, and b and c integer vectors. If* $\max\{c^\top x \mid Ax \le b, x \in \mathbb{R}_+^n\}$ and its dual $\min\{b^\top y \mid A^\top y \ge c, y \in \mathbb{R}_+^m\}$ have optimal solutions, then the optimal solutions must be at integer vertices.
Theorem: If for any integer vector b, the vertices of polyhedron $P = \{x \in \mathbb{R}_+^n \mid Ax \le b\}$ are integer points, then A is a totally unimodular matrix.
Take any $k \times k$ non-singular submatrix A_1 of A. Since (A, I) is row complete, there exists an $m \times m$ non-singular submatrix of (A, I) of the form $\tilde{A} = \begin{pmatrix}A_1 & 0\\A_2 & I_{m-k}\end{pmatrix}$. Let $b = \tilde{A}z + e_i$, where $z \in \mathbb{Z}^m$. Then $\tilde{A}^{-1}b = z + \tilde{a}_i^{-1}$, where $\tilde{a}_i^{-1}$ represents the ith column of $\tilde{A}^{-1}$. By appropriate choice of z such that $z + \tilde{a}_i^{-1} \ge 0$, it is clear that $z + \tilde{a}_i^{-1}$ is the direction vector formed by the basic variable components of the vertices of P. By assumption, $z + \tilde{a}_i^{-1} \in \mathbb{Z}^m$, hence $\tilde{a}_i^{-1} \in \mathbb{Z}^m$. Thus, $\tilde{A}^{-1}$ is an integer matrix, and therefore A_1^{-1} is an integer matrix. Since A_1 and A_1^{-1} are integer matrices, $\det(A_1)$ and $\det(A_1^{-1})$ are integers. Moreover,
$|\det(A_1)| \cdot |\det(A_1^{-1})| = |\det(A_1A_1^{-1})| = 1$,
hence $|\det(A_1)| = 1$. Thus, matrix A is totally unimodular. □
Corollary: A is totally unimodular iff for integer vectors a, b, c, d, the vertices of polyhedron $\{x \mid a \le x \le b, c \le Ax \le d\}$ are integer points.

8.2.2 Bipartite Graph

Undirected graph $G = (V, E)$, define the incidence matrix M of G, where the rows and columns are labeled by the vertex set V and edge set E respectively. If e is incident to vertex v, then $M_{v,e} = 1$; otherwise, $M_{v,e} = 0$.
If the vertex set V of a graph $G = (V, E)$ can be partitioned into two non-empty subsets V_1 and V_2 such that the two endpoints of each edge belong to V_1 and V_2 respectively, then the graph is called a bipartite graph.
Theorem: M is totally unimodular iff graph G is a bipartite graph.
Independent set: a set of vertices in a graph, no two of which are adjacent.
Edge cover: a set of edges such that every vertex of the graph is on at least one edge of the set.
Matching M: a set of edges without common vertices.
Vertex cover R: a set of vertices that includes at least one endpoint of every edge. $|M| \le |R|$
Let M be the incidence matrix of the bipartite graph $G = (V, E)$, we obtain $\max\{e^\top y \mid My \le e, y \in \mathbb{Z}_+^{|V|}\} = \min\{e^\top x \mid M^\top x \ge e, x \in \mathbb{Z}_+^{|E|}\}$.
Suppose each vertex in graph is incident to at least one edge, **the max number of independent vertices = the min number of edges in a edge cover.**
Similarly, by total unimodularity of the incidence matrix of a bipartite graph, $\max\{e^\top x \mid M^\top x \le e, x \in \mathbb{Z}_+^{|E|}\} = \min\{e^\top y \mid My \ge e, y \in \mathbb{Z}_+^{|V|}\}$.
The above equation shows that **the maximum number of edges in a matching = the minimum number of vertices in a vertex cover.**

8.3 The Transportation and assignment problems

8.3.1 Matching and Assignment on Graph $G = (V, E)$

Definition: A subset M of the edges of G is called a **matching** if no two edges in M share a common vertex (any vertex of G is incident to at most one edge of M). If vertex u is incident to any edges in M, then u is **M-exposed**.
Definition: Given a subset of vertices $R \subseteq V$, if every edge in E has at least one vertex in R, then R is called a **vertex cover** of G.
Definition: If every vertex in G is matched by an edge in M, then M is called a **perfect matching**. If there is no other matching M' such that $|M'| > |M|$, then M is called a **maximum matching**. Clearly, every perfect matching is a maximum matching.
Theorem: Given a graph G, for any matching M and any vertex cover R, it always holds that $|M| \le |R|$.
Definition: Given a matching M in G, if a path P alternates between edges in M and $E \setminus M$, then P is called an **alternating path** w.r.t. M. If the start

and end vertices of P are not adjacent with edges in M, then P is called an **augmenting path** w.r.t. M.

Theorem: M is max matching iff there're no augmenting paths w.r.t. M.

Maximum Matching Algorithm:

0. Given a bipartite graph $G = (V_1, V_2, E)$, let M be a matching of G. All vertices are unmarked and unchecked.
1. 1.0 Mark all M-**un**exposed vertices in V_1 with $*$.
   - 1.1 If all marked vertices are checked, go to Step 3. Otherwise, select a marked unchecked vertex i. If $i \in V_1$, go to Step 1.2. If $i \in V_2$, go to Step 1.3.
   - 1.2 Check all edges (i, j) incident to $i \in V_1$. If $(i, j) \in E \setminus M$ and j is unmarked, mark j as i. Go to Step 1.1.
   - 1.3 Check all edges (j, i) incident to $i \in V_2$. If i is M-**un**exposed, go to Step 2. Otherwise, find edge $(j, i) \in M$ where j is unmarked, mark j as i. Go to Step 1.1.
2. Starting from $i \in V_2$, use the marked vertices to find an augmenting path P. Let $M := (M \cup P) \setminus (M \cap P)$. Remove all marks and go to Step 1.
3. Let V_1^+ and V_2^+ denote the marked vertices in V_1 and V_2, V_1^- and V_2^- denote the unmarked vertices. The algorithm **terminates**.

Result: $R = V_1^- \cup V_2^+$ is a cover of G; $|M| = |R|$, and M is max matching.

Assignment Problem:

Suppose a company is preparing to assign n workers $X_1, \dots, X_n$ to n tasks $Y_1, \dots, Y_n$. Consider a weighted complete bipartite graph $G = (V_1, V_2, E)$, where $V_1 = \{X_1, \dots, X_n\}$, $V_2 = \{Y_1, \dots, Y_n\}$, and weight c_{ij} on edge (X_i, Y_j) represents the efficiency of worker X_i completing task Y_j. Equivalent to finding a perfect matching with max weight.

$$\begin{aligned} \max \quad & \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{j=1}^n x_{ij} = 1, \quad i = 1, 2, \dots, n, \\ & \sum_{i=1}^n x_{ij} = 1, \quad j = 1, 2, \dots, n, \\ & x_{ij} \in \{0, 1\}, \quad i, j = 1, 2, \dots, n. \end{aligned}$$

Bipartite graph totally unimodular, the last constraint relaxed:
$x_{ij} \geq 0, \quad i, j = 1, 2, \dots, n.$

Dual problem: $\begin{aligned} \min \quad & \sum_{i=1}^n u_i + \sum_{j=1}^n v_j \\ \text{s.t.} \quad & u_i + v_j \geq c_{ij}, \quad i, j = 1, 2, \dots, n. \end{aligned}$

Theorem 4.8 By the LP duality theorem, if there exists a feasible solution x^* to the assignment problem and a pair u, v that satisfy the following two conditions: (i) $\bar{c}_{ij} = c_{ij} - u_i - v_j \leq 0$; (ii) When $x_{ij}^* = 1$, $\bar{c}_{ij} = 0$, then x^* is the optimal assignment solution, and the optimal value is $\sum_{i=1}^n u_i + \sum_{j=1}^n v_j$.

Assignment Problem Algorithm:

0. Given an initial u, v satisfying $\bar{c}_{ij} \leq 0$, $i, j = 1, 2, \dots, n$. Let $\bar{E} = \{(i, j) \in E \mid \bar{c}_{ij} = 0\}$. Use Algorithm to find the maximum matching M^* in bipartite graph $\bar{G} = (V_1, V_2, \bar{E})$. If $|M^*| = n$, the algorithm **ends** and M^* is the optimal assignment solution. Otherwise, record $M = M^*$ and the marked vertex sets V_1^+, V_2^+. Go to Step 2.
1. (Original). Let $\bar{E} = \{(i, j) \in E \mid \bar{c}_{ij} = 0\}$. Based on the matched M and marked vertex sets V_1^+, V_2^+, continue to find the maximum matching M^* in $\bar{G} = (V_1, V_2, \bar{E})$. If $|M^*| = n$, the algorithm **terminates** and M^* is the optimal assignment solution. Otherwise, record $M = M^*$ and V_1^+, V_2^+, then go to Step 2.
2. (Dual). let $\delta = \min\{-\bar{c}_{ij} \mid i \in V_1^+, j \in V_2 \setminus V_2^+\}$; for all $i \in V_1^+$, let $u_i := u_i - \delta$; for all $j \in V_2^+$, let $v_j := v_j + \delta$. Go to Step 1.

8.3.2 Transportation Problem

Problem: A certain product has m production sites, denoted as $A_1, \dots, A_m$, with production quantities $a_1, \dots, a_m$. There are n sales sites, denoted as $B_1, \dots, B_n$, with sales quantities $b_1, \dots, b_n$. Suppose the unit transportation cost from production site A_i to sales site B_j is c_{ij}. How should these products be transported to minimize the total cost?

$$\begin{aligned} \min \quad & \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} \quad \text{s.t.} \quad \sum_{j=1}^n x_{ij} = a_i, \quad \forall 1 \leq i \leq m \\ & \sum_{i=1}^m x_{ij} = b_j, \quad \forall 1 \leq j \leq n \\ & x_{ij} \geq 0, \quad \forall i = 1, 2, \dots, m, \ j = 1, 2, \dots, n \end{aligned}$$

Assume $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j = Q$ (Balanced transportation problem)

It's a special case of minimum cost network flow problem.

Dual problem: $\max \quad \sum_{i=1}^m a_i u_i + \sum_{j=1}^n b_j v_j \quad \text{s.t.} \quad u_i + v_j \leq c_{ij}$

$\Rightarrow c_{ij} - (u_i + v_j) \geq 0 \ (i = 1, 2, \dots, m, j = 1, 2, \dots, n)$

Algorithm:

1. Find the Minimum Spanning Tree (The basic feasible solution corresponds to a spanning tree; find the arcs with the minimum costs first).
2. Find a dual solution by solving $c_{ij} = u_i + v_j$ and let $v_n = 0$.
3. Calculate Verification Number: $\sigma_{ij} = c_{ij} - (u_i + v_j)$, $\forall i, j$. If all σ_{ij}'s are non-negative, then **terminate**; otherwise go to step 4.
4. Let $\bar{w}_{st} = \min_{i,j}\{\bar{w}_{ij} | \bar{w}_{ij} < 0\}$. Add arc (s, t) into the spanning tree found in step 1 and let $x_{st} = \theta$ to remove an arc. Get an improved spanning tree, go to step 2.

8.4 Graph and Network Flows

8.4.1 Shortest Path Problem

Directed graph $D = (V, A)$, let $c_{u,v}$ represent the weight on arc $(u, v) \in A$ (length of the arc). Solve the shortest $s - t$ path problem:

$$\begin{aligned} \min \quad & \sum_{(u,v)\in A} c_{u,v} x_{u,v} \\ \text{s.t.} \quad & \sum_{u\in V^+(v)} x_{v,u} - \sum_{u\in V^-(v)} x_{u,v} = 1, v = s, \\ & \sum_{u\in V^+(v)} x_{v,u} - \sum_{u\in V^-(v)} x_{u,v} = 0, \forall v \in V \setminus \{s, t\}, \\ & \sum_{u\in V^+(v)} x_{v,u} - \sum_{u\in V^-(v)} x_{u,v} = -1, v = t, \\ & x \in \mathbb{Z}_+^{|A|}. \end{aligned}$$

By total unimodularity, LP relaxation of problem has integral optimal solution.

Dual Problem: $\begin{aligned} \max \quad & \pi_t - \pi_s \\ \text{s.t.} \quad & \pi_v - \pi_u \leq c_{u,v}, \forall (u, v) \in A. \end{aligned}$

Theorem: z is length of shortest $s - t$ path iff there exists $\pi = (\pi_v)_{v\in V}$ satisfying $\pi_s = 0$, $\pi_t = z$, $\pi_v - \pi_u \leq c_{u,v}$, where $(u, v) \in A$.

Dijkstra Algorithm:

```
for all u ∈ V:
   dist(u) = ∞
   prev(u) = nil
dist(s) = 0
H = makequeue(V)
while H is not empty:
```

```
u = deletemin(H)
for all edges (u,v) ∈ E:
   if dist(v) > dist(u) + l(u,v):
      dist(v) = dist(u) + l(u,v)
      prev(v) = u
      decreasekey(H,v)
```

Bellman-Ford Algorithm (allowing negative length):

```
for all u ∈ V:
   dist(u) = ∞
   prev(u) = nil
```

```
dist(s) = 0
repeat |V| - 1 times:
   for all e ∈ E:
      dist(v) = min{dist(v), dist(u) + l(u,v)}
```

Floyd-Warshall Algorithm (all-pairs shortest paths):

```
for i = 1 to n:
  for j = 1 to n:
    dist(i,j,0) = ∞
for all (i,j) ∈ E:
  dist(i,j,0) = ℓ(i,j)
```

```
for k = 1 to n:
  for i = 1 to n:
    for j = 1 to n:
      dist(i,j,k) = min{dist(i,k,k-1)
                        + dist(k,j,k-1), dist(i,j,k-1)}
```

8.4.2 Maximum Flow-Minimum Cut Problem

Incidence matrix: Directed graph $D = (V, A)$, where V represents set of vertices and A represents set of arcs, $(u, v) \in A$ indicates an arc from u to v. Let M be the $V \times A$ incidence matrix of the graph. If a flows into v, then $M_{v,a} = 1$; if a flows out of v, then $M_{v,a} = -1$; otherwise, $M_{v,a} = 0$.

Theorem: The incidence matrix M of a $D = (V, A)$ is totally unimodular.

Circulation: For any $x \in \mathbb{R}_+^{|A|}$ that satisfies $Mx = 0$, the following holds for any v: $\sum_{u\in V^-(v)} x_{u,v} = \sum_{u\in V^+(v)} x_{v,u}$, where x can be viewed as a circulation in the graph M, and the inflow and outflow at each vertex are equal.

Since M is totally unimodular, for any integer vector $c \in \mathbb{Z}_+^{|A|}$, polyhedron $P = \{x \mid Mx = 0, 0 \leq x \leq c\}$ has integer vertices. Therefore, if there is a circulation x s.t. $0 \leq x \leq c$, then there must exist an integer circulation.

Circulation Problem: Let $f_{u,v}$ denote the profit per unit flow on arc (u, v). Then the **maximum profit circulation problem** with capacity constraints can be formulated as: $\max\{f^\top x \mid Mx = 0, x \leq c, x \in \mathbb{R}_+^{|A|}\}$, i.e.,

$$-\min \langle [-f, 0], [x, s] \rangle, \quad \text{s.t.} \begin{bmatrix} M^\top & O \\ I & I \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ c \end{bmatrix}.$$

Its **dual problem** is: $\min\{c^\top y \mid M^\top z + y \geq f, \ y \in \mathbb{R}_+^A, \ z \in \mathbb{R}^{|V|}\}$.

By Corollary, if above two problems have optimal solutions, they must attain optimal values at integer vertices, and their objective function values are equal.

Maximum Flow Problem: Suppose for a directed graph there exists an arc (t, s) with unit profit $f_{t,s} = 1$, and other arcs (u, v) have $f_{u,v} = 0$. If $c_{t,s} = +\infty$, problem **reformulated**:

$$\begin{aligned} \max \quad & x_{t,s} \\ \text{s.t.} \quad & \sum_{u\in V^-(v)} x_{u,v} - \sum_{u\in V^+(v)} x_{v,u} = 0, \quad \forall v \in V, \\ & 0 \leq x_{u,v} \leq c_{u,v}, \quad \forall (u, v) \in A. \end{aligned}$$

Consider $D' = (V, A)$, where A does not contain arc (t, s). The above problem can be seen as the maximum flow problem from s to t in the graph D', with arc (t, s) added artificially. The **dual problem** of above formulated:

$$\begin{aligned} \min \quad & \sum_{(u,v)\neq(t,s)} c_{u,v} y_{u,v} \\ \text{s.t.} \quad & z_u - z_v \leq y_{u,v}, \quad \forall (u, v) \neq (t, s) \text{ or } \forall (u, v) \in A, \\ & z_t \geq z_s + 1. \end{aligned}$$

Let $U = \{v \in V \mid z_v < z_t\}$ and $\bar{U} = V \setminus U = \{v \in V \mid z_v \geq z_t\}$. For any arc $(u, v) \in A$, if $u \in U$ and $v \in \bar{U}$, then $y_{u,v} \geq z_v - z_u \geq z_t - z_s \geq 1$. Thus, the following holds:

$\sum_{(u,v)\in A} c_{u,v} y_{u,v} \geq \sum_{(u,v)\in A, u\in U, v\in\bar{U}} c_{u,v} y_{u,v} \geq \sum_{\dots} c_{u,v}$.

Construct a feasible solution $\hat{y}$ as follows: when $(u, v) \in A$ and $u \in U$, $v \in \bar{U}$, set $\hat{y}_{u,v} = 1$; otherwise, set $\hat{y}_{u,v} = 0$. Therefore, $\hat{y}$ is an optimal 0-1 solution to problem. Thus, problem can be viewed as the minimum $s - t$ cut problem in the directed graph D'.

$\min_U \left\{ \sum_{(u,v)\in A, u\in U, v\in\bar{U}} c_{u,v} \mid s \in U, t \in \bar{U} \right\}$.

Theorem (with constraints,) The maximum $s - t$ flow problem and the minimum $s - t$ cut problem are dual to each other and optimal values are equal.

Algorithm: Start with zero flow. residual network RN=N

Repeat: choose an appropriate path from s to t, and increase flow along the edges of this path as much as possible. update the residual network RN

$G^f = (V, E^f)$ **until no s-t path**, where residual capacities c^f:

$$c_{uv}^f = \begin{cases} c_{uv} - f_{uv} & \text{if } (u, v) \in E \text{ and } f_{uv} < c_{uv} \\ f_{vu} & \text{if } (v, u) \in E \text{ and } f_{vu} > 0 \end{cases}$$

8.4.3 Minimum Cost Network Flow Problem

Problem: Given a directed graph $D = (V, A)$, let $h_{u,v}$ represent the maximum capacity of arc (u, v), b_v represent the demand at vertex v, and $c_{u,v}$ represent the cost per unit flow on arc (u, v). Let

$V^+(v) = \{u \in V \mid (v, u) \in A\}$, $\quad V^-(v) = \{u \in V \mid (u, v) \in A\}$.

Then the minimum cost network flow problem can be formulated as

$$\begin{aligned} \min \quad & \sum_{(u,v)\in A} c_{u,v} x_{u,v} \\ \text{s.t.} \quad & \sum_{u\in V^+(v)} x_{v,u} - \sum_{u\in V^-(v)} x_{u,v} = b_v, \quad \forall v \in V, \\ & 0 \leq x_{u,v} \leq h_{u,v}, \quad \forall (u, v) \in A. \end{aligned}$$

The above problem can be formulated $\min\{c^\top x \mid Mx = b, \ 0 \leq x \leq h\}$.

If the problem is feasible, the sum of demands must be 0, i.e., $\sum_{v\in V} b_v = 0$.

If the capacities $h_{u,v}$ and demands b_v are integers, by the total unimodularity of M, the problem has an integer optimal solution.

Algorithm for Min-Cost-Max-Flow Problem:

1. Set the initial feasible flow to zero.
2. According to step $k - 1$, obtain the minimum cost flow $f^{(k-1)}$, and construct the graph $L(f^{(k-1)})$ (similar to RN, but with cost as weight).
3. In $L(f^{(k-1)})$, find the shortest path from v_s to v_t. If no shortest path exists, then $f^{(k-1)}$ is the minimum cost maximum flow, and **terminates**.
If a shortest path exists, get the corresponding augmenting chain in the graph $f^{(k-1)}$, and proceed to step 4.
4. On the augmenting chain, adjust $f^{(k-1)}$ with the adjustment quantity θ:

$$\theta = \min\left[\min_{(v_i,v_j)\in\mu^+}\left(c_{ij} - f_{ij}^{(k-1)}\right), \min_{(v_i,v_j)\in\mu^-}\left(f_{ij}^{(k-1)}\right)\right]$$

$$f_{ij}^{(k)} = \begin{cases} f_{ij}^{(k-1)} + \theta & (v_i, v_j) \in \mu^+ \\ f_{ij}^{(k-1)} - \theta & (v_i, v_j) \in \mu^- \\ f_{ij}^{(k-1)} & (v_i, v_j) \notin \mu \end{cases}$$

5. Repeat the above steps for $f^{(k)}$, returning to step 2.

8.5 Dynamic Programming

8.5.1 Shortest Path and Principle of Optimality

Property: Suppose the shortest path from s to t passes through node p. Then the sub-paths (s, p) and (p, t) are the shortest paths from s to p and from p to t, respectively.

Property: Let $d(v)$ be the shortest path from s to v, then $d(v) = \min_{i\in V^-(v)}\{d(i) + c_{iv}\}$, where $V^-(v)$ denotes the set of all nodes that can reach node v directly.

Let $D_k(i)$ represent the shortest path from s to i that contains at most k isolated paths. It can be derived by the following recursive formula:
$D_k(j) = \min\{D_{k-1}(j), \min_{i\in V^-(j)}[D_{k-1}(i) + c_{ij}]\}$.

Principle of Optimality: An optimal decision for a multi-stage decision problem is one in which each stage's decision is also optimal.

8.5.2 Knapsack Problem (13Q3)

0-1 Knapsack Problem:

There is a knapsack with a capacity of b. Consider n items, where the weight of item j is a_j and the value is c_j. How to choose items to maximize the total value in the knapsack?

Let $x_j = 1$ if item j is selected, 0 otherwise.

$f^* = \max\left\{\sum_{j=1}^n c_j x_j \middle| \sum_{j=1}^n a_j x_j \leq b, x \in \{0, 1\}^n\right\}$,

where $n, b, c_j, a_j, j = 1, \dots, n$ are positive integers.

Algorithm: The knapsack problem can be seen as selecting the items from 1 to n in sequence. For $k = 1, \dots, n$, and $\lambda = 0, 1, \dots, b$, define

$f_k(\lambda) = \max\left\{\sum_{j=1}^k c_j x_j \middle| \sum_{j=1}^k a_j x_j \leq \lambda, x \in \{0, 1\}^k\right\}$.

It is easy to see that $f^* = f_n(b)$.

We can use the recursive method to find f_n, that is, use f_{n-1} to calculate f_n, use f_{n-2} to calculate f_{n-1}, and so on. The initial condition for recursion is

$f_1(\lambda) = \begin{cases} c_1, & \text{if } a_1 \leq \lambda, \\ 0, & \text{if } a_1 > \lambda. \end{cases}$

When $\lambda \geq 0$, $f_0(\lambda) = 0$. For $k = 2, \dots, n$, $\lambda = 0, 1, \dots, b$,

$f_k(\lambda) = \begin{cases} f_{k-1}(\lambda), & \text{if } a_k > \lambda, \\ \max\{f_{k-1}(\lambda), c_k + f_{k-1}(\lambda - a_k)\}, & \text{if } a_k \leq \lambda. \end{cases}$

Backtrack to find the optimal solution:

$x_n^* = \begin{cases} 0, & f_n(b) = f_{n-1}(b), \\ 1, & \text{otherwise}. \end{cases}$

Let $\lambda_k^* = b - \sum_{j=k+1}^n a_j x_j^*$, then for $k = n - 1, \dots, 1$, we have

$x_k^* = \begin{cases} 0, & f_k(\lambda_k^*) = f_{k-1}(\lambda_k^*), \\ 1, & \text{otherwise}. \end{cases}$

Linear Integer Knapsack Problem (with repetition):

$z^* = \max\left\{\sum_{j=1}^n c_j x_j \middle| \sum_{j=1}^n a_j x_j \leq b, x \in \mathbb{Z}_+^n\right\}$,

Method 1:

$g_r(\lambda) = \max\left\{\sum_{j=1}^r c_j x_j \middle| \sum_{j=1}^r a_j x_j \leq \lambda, x \in \mathbb{Z}_+^r\right\}$.

It is easy to see $z^* = g_n(b)$. $g_r(\lambda) = \max\{g_{r-1}(\lambda), c_r + g_r(\lambda - a_r)\}$.

Backtrack to find the optimal solution:

$p_r(\lambda) = \begin{cases} 0, & g_r(\lambda) = g_{r-1}(\lambda), \\ 1 + p_r(\lambda - a_r), & \text{otherwise}. \end{cases}$

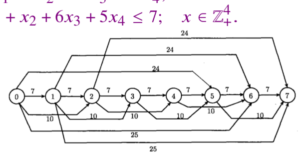
$x_n^* = p_n(b)$.

Let $\lambda_k^* = b - \sum_{j=k+1}^n a_j x_j^*$, then for $k = n - 1, \dots, 1$, $x_k^* = p_k(\lambda_k^*)$.

Method 2: For $\lambda = 0$ to b: $z(\lambda) = \max_{j: a_j \leq \lambda} z(\lambda - a_j) + c_j$.

Method 3: Consider it as a longest path problem. Construct a DAG $D = (V, A)$, with vertices $0, 1, \dots, b$, and arcs $(\lambda, \lambda + a_j)$, $\lambda \in \mathbb{Z}_+$, $\lambda \leq b - a_j$ $(j = 1, \dots, n)$, with weights c_j, and arcs $(\lambda, \lambda + 1)$, $\lambda \in \mathbb{Z}_+$, $\lambda \leq b - 1$, with weight 0. Then $z(\lambda)$ is the longest path from vertex 0 to vertex λ. Figure gives a directed graph for the following example:

$$\begin{aligned} \max \quad & 10x_1 + 7x_2 + 25x_3 + 24x_4, \\ \text{s.t.} \quad & 2x_1 + x_2 + 6x_3 + 5x_4 \leq 7; \quad x \in \mathbb{Z}_+^4. \end{aligned}$$



paths with zero length are omitted.

8.5.3 Investment Allocation Problem:

With \$10W capital, invest in three projects with benefits given by $g_1(x_1) = 4x_1$, $g_2(x_2) = 9x_2$, $g_3(x_3) = 2x_3^2$. How should the investment amounts be allocated?

$$\begin{aligned} \max \quad & 4x_1 + 9x_2 + 2x_3^2 \quad \text{s.t.} \quad x_1 + x_2 + x_3 = 10 \\ & x_i \geq 0, \quad i = 1, 2, 3 \end{aligned}$$

State: At stage k, the available amount s_k, with $s_1 = 10$.

Decision: At stage k, the actual investment amount $x_k = u_k(s_k)$.

Allowed Decision Set: $0 \leq x_k \leq s_k$.

Stage Objective: $4x_1$, $9x_2$, $2x_3^2$.

State Transition Equation: $s_{k+1} = s_k - x_k$.

Backward Solution:

$f_k(s_k) = \max_{0\leq x_k\leq s_k}\{d_k(s_k, x_k) + f_{k+1}(T_k(s_k, x_k))\}$.

$f_4(s_4) = 0$

$f_3(s_3) = \max_{0\leq x_3\leq s_3} 2x_3^2 \Rightarrow u_3^*(s_3) = s_3$, $f_3(s_3) = 2(s_2 - x_2)^2$

$f_2(s_2) = \max_{0\leq x_2\leq s_2}\left(9x_2 + 2(s_2 - x_2)^2\right)$

Since the objective function is convex, $f_2(s_2) = \max\{2s_2^2, 9s_2\}$.

Thus, if $s_2 \geq \frac{9}{2}$, then $u_2^*(s_2) = 0$, $f_2(s_2) = 2s_2^2$;

if $s_2 < \frac{9}{2}$, then $u_2^*(s_2) = s_2$, $f_2(s_2) = 9s_2$

$f_1(10) = \max_{0\leq x_1\leq 10}(4x_1 + f_2(10 - x_1))$, where

$f_2(10 - x_1) = \begin{cases} 2(10 - x_1)^2, & \text{if } x_1 \leq \frac{11}{2} \\ 9(10 - x_1), & \text{if } x_1 > \frac{11}{2} \end{cases}$

For $0 \leq x_1 \leq \frac{11}{2}$, the objective function is convex, so

$f_1(10) = \max\left\{2 \times 10^2, \ 4 \times \frac{11}{2} + 2 \times \left(10 - \frac{11}{2}\right)^2\right\} = 200$

For $\frac{11}{2} < x_1 \leq 10$, the objective function is $90 - 5x_1$, so

$u_1^*(10) = \frac{11}{2}$, $\quad f_1(10) = 90 - 5 \times \frac{11}{2} = 62.5$

Combining both cases, we find $u_1^*(10) = 0$, $\quad f_1(10) = 200$.

Result: $x_1^* = u_1^*(10) = 0$, $f_1(10) = 200$, $\quad s_2^* = 10 - x_1^* = 10$

$x_2^* = u_2^*(10) = 0$, $f_2(10) = 2 \times 10^2 = 200$, $\quad s_3^* = 10 - x_2^* = 10$

$x_3^* = u_3^*(10) = 10$, $f_3(10) = 2 \times 10^2 = 200$

8.5.4 TSP

A traveling salesman needs to visit n cities to sell his products. He must visit each city exactly once and return to the starting city. The distance between each pair of cities is known (if direct travel is not possible, the distance is set to $+\infty$). How to choose the travel route to minimize the total travel distance?

Let c_{ij} be the distance from city i to city j. Let

$x_{ij} = \begin{cases} 1, & \text{if the route includes traveling directly from city } i \text{ to city } j, \\ 0, & \text{otherwise}. \end{cases}$

$$\begin{aligned} \min \quad & \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{j\neq i} x_{ij} = 1, \quad i = 1, \dots, n, \\ & \sum_{i\neq j} x_{ij} = 1, \quad j = 1, \dots, n, \qquad x \in \{0, 1\}^n \\ & \sum_{i\in S} \sum_{j\in S} x_{ij} \leq |S| - 1, \quad \forall S \subset N, \ 2 \leq |S| \leq n - 1 \end{aligned}$$

8.5.5 Stochastic Shortest Path Problem

Five cities are interconnected with roads, with round-trip distances being the same. Find the shortest path from each city to the 5-th city.
Define the optimal path from each point to the destination: $f(v_i)$, $1 \le i \le 5$.
If the optimal path exists, should satisfy:
$f(v_i) = \min_{1 \le j \le 5}\{c_{ij} + f(v_j)\}, \forall 1 \le i \le 5,$
where c_{ij} represents the direct distance between v_i and v_j ($c_{ii} = 0$).
Functional Space Iteration Method (Value Iteration):
First, take $f_1(v_i) = c_{i5}, \quad 1 \le i \le 5.$
Then substitute according to the following formula:
$f_{k+1}(v_i) = \min_{1 \le j \le 5}\{c_{ij} + f_k(v_j)\}, \quad \forall k \ge 1$
If for some k, $f_{k+1}(v_i) = f_k(v_i), \quad \forall 1 \le i \le 5$, then
$f_k(v_i) = \min_{1 \le j \le 5}\{c_{ij} + f_k(v_j)\}, \quad \forall 1 \le i \le 5$
Thus, we get $f(v_i) = f_k(v_i), \quad \forall 1 \le i \le 5.$

8.5.6 Chain matrix multiplication in $O(n^3)$ (12Q3)

The DAG looks like pyramid, $|V| = O(n^2)$, $|E| = O(n^3)$.
Define subproblem $C(i, j)$ = min cost multiplying $A_i \times A_{i+1} \times \ldots \times A_j$.
The first branch in subtree will split the product in two pieces: $A_i \times \ldots \times A_k$ and $A_{k+1} \times \ldots \times A_j$, for some $i \le k \le j$. The cost of the subtree is the cost of these two partial products, plus the cost of combining them.

```
for i = 1 to n:   C(i,i) = 0
for s = 1 to n-1:
  for i = 1 to n-s:
    j = i + s
    C(i,j) = min{C(i,k) + C(k+1,j) + m_{i-1} · m_k · m_j : i ≤ k < j}
return C(1,n)
```

8.6 Branch-and-Bound Algorithm

(i) If one of the linear programs LP_i is infeasible, i.e., $P_i = \emptyset$, then we also have $S_i = \emptyset$ since $S_i \subseteq P_i$. Thus MILP_i is infeasible and does not need to be considered any further. We say that this problem is *pruned by infeasibility*.

(ii) Let (x^i, y^i) be an optimal solution of LP_i and z_i its value, $i = 1, 2$.
  (iia) If x^i is an integral vector, then (x^i, y^i) is an optimal solution of MILP_i and a feasible solution of MILP. Problem MILP_i is solved, and we say that it is *pruned by integrality*. Since $S_i \subseteq S$, it follows that $z_i \le z^*$, that is, z_i is a lower bound on the value of MILP.
  (iib) If x^i is not an integral vector and z_i is smaller than or equal to the best known lower bound on the value of MILP, then S_i cannot contain a better solution and the problem is *pruned by bound*.
  (iic) If x^i is not an integral vector and z_i is greater than the best known lower bound, then S_i may still contain an optimal solution to MILP. Let x_i^j be a fractional component of vector x^i. Let $f' := x_i^j$,
$S_{i1} := S_i \cap \{(x, y) : x_j \le f'\}, S_{i2} := S_i \cap \{(x, y) : x_j > f'\}$
and repeat the above process.

8.7 The Cutting Plane Method

The idea is to find an inequality $ax + by \le \beta$ that is satisfied by every point in S and such that $ax^0 + by^0 > \beta$. The existence of such an inequality is guaranteed when (x^0, y^0) is a basic solution of (1.6).
An inequality $av \le \beta$ is valid for a set $K \subseteq \mathbb{R}^d$ if it is satisfied by every point $v \in K$. A valid inequality $ax + by \le \beta$ for S that is violated by (x^0, y^0) is a *cutting plane* separating (x^0, y^0) from S. Let $ax + by \le \beta$ be a cutting plane and define $P_1 := P_0 \cap \{(x, y) : ax + by \le \beta\}$.
Since $S \subseteq P_1 \subseteq P_0$, the linear programming relaxation of MILP based on P_1 is stronger than the natural linear programming relaxation (1.5), in the sense that the optimal value of the linear program $\max\{cx + hy : (x, y) \in P_1\}$ is at least as good an upper-bound on the value z^* as z_0, while the optimal solution (x^0, y^0) of the natural linear programming relaxation does not belong to P_1. The recursive application of this idea leads to the *cutting plane approach*.

9 Numerical Methods

9.1 Descent methods

- $x^{(k+1)} = x^{(k)} + t^{(k)} \Delta x^{(k)}$
- the search direction in a descent method must satisfy $\nabla f(x^{(k)})^T \Delta x^{(k)} < 0$
- **Exact line search:** $t = \arg\min_{s \ge 0} f(x + s\Delta x)$
- **Backtracking line search:** Begin from $t = 1$ and reduce t by a factor $\beta \in (0, 1)$ until $f(x + t\Delta x) \le f(x) + \alpha t \nabla f(x)^\top \Delta x$ for some $\alpha \in (0, 0.5)$.

9.2 Steepest Descent

- For any norm $\|\cdot\|$ on $\mathbb{R}^n$, define a **normalized steepest descent direction** as
$$\Delta x_{nsd} = \arg\min\{\nabla f(x)^\top v \mid \|v\| \le 1\}$$
- Define **unnormalized steepest descent** As
$$\Delta x_{sd} = \|\nabla f(x)\|_* \Delta x_{nsd}$$

Algorithm 9.4 *Steepest descent method.*

given a starting point $x \in \textbf{dom} f$.
repeat
1. Compute steepest descent direction Δx_{sd}.
2. *Line search.* Choose t via backtracking or exact line search.
3. *Update.* $x := x + t\Delta x_{\text{sd}}$.

until stopping criterion is satisfied.

9.3 Newton's Method

Newton's Method: $x_{k+1} = x_k - \nabla^2 f(x_k)^{-1} \nabla f(x_k), \quad k = 1, 2, \ldots.$

9.4 Which set is convex?

- (a) A slab $\{x \in \mathbb{R}^n \mid \alpha \le a^T x \le \beta\}$ is convex because it is the intersection of two halfspaces, both of which are convex. The intersection of convex sets is convex.
- (b) A rectangle $\{x \in \mathbb{R}^n \mid \alpha_i \le x_i \le \beta_i,\ i = 1, \ldots, n\}$ is convex because it is a finite intersection of halfspaces. It is also a polyhedron.
- (c) A wedge $\{x \in \mathbb{R}^n \mid a_1^T x \le b_1,\ a_2^T x \le b_2\}$ is convex because it is the intersection of two halfspaces, which are convex. (If $b_1 = 0$ and $b_2 = 0$, it is a cone.)
- (d) The set $\{x \mid \|x - x_0\|_2 \le \|x - y\|_2,\ \forall y \in S\}$ is convex because for each $y \in S$, the set $\{x \mid \|x - x_0\|_2 \le \|x - y\|_2\}$ is a halfspace, and the intersection of halfspaces (over all y) is convex.
- (e) The set $\{x \mid \text{dist}(x, S) \le \text{dist}(x, T)\}$ is **not** convex in general. For example, with $S = \{-1, 1\}$ and $T = \{0\}$ in $\mathbb{R}$, the resulting set is $\{x \mid x \le -1/2 \text{ or } x \ge 1/2\}$, which is not convex.
- (f) The set $\{x \mid x + S_2 \subseteq S_1\}$ (with S_1 convex) is convex because it can be written as $\bigcap_{y \in S_2}(S_1 - y)$, an intersection of convex sets, which is convex.
- (g) The set $\{x \mid \|x - a\|_2 \le \theta\|x - b\|_2\}$ with $a \ne b$ and $0 \le \theta \le 1$ is convex. If $\theta = 1$, it is a halfspace; if $\theta < 1$, it is a ball, both of which are convex sets.
- (a) $\alpha \le \mathbb{E} f(x) \le \beta$ is convex because the constraint is linear in p: $\sum_{i=1}^n p_i f(a_i)$.
- (b) $\text{prob}(x \ge \alpha) \le \beta$ is convex because it is a linear constraint: $\sum_{i:a_i \ge \alpha} p_i \le \beta$.
- (c) $\mathbb{E}|x^3| \le \alpha\mathbb{E}|x|$ is convex because it is a linear constraint: $\sum_{i=1}^n p_i(|a_i|^3 - \alpha|a_i|) \le 0$.
- (d) $\mathbb{E}x^2 \le \alpha$ is convex because it is a linear constraint: $\sum_{i=1}^n p_i a_i^2 \le \alpha$.
- (e) $\mathbb{E}x^2 \ge \alpha$ is convex because it is a linear constraint: $\sum_{i=1}^n p_i a_i^2 \ge \alpha$.
- (f) $\text{var}(x) \le \alpha$ is **not convex** in p because variance $\text{var}(x) = \mathbb{E}x^2 - (\mathbb{E}x)^2$ is not a convex function of p. A counterexample exists for $n = 2$.
- (g) $\text{var}(x) \ge \alpha$ is convex because the superlevel set of the quadratic function $\mathbb{E}x^2 - (\mathbb{E}x)^2$ is convex (since the matrix $A = aa^T$ is positive semidefinite).
- (h) $\text{quartile}(x) \ge \alpha$ is convex because it is equivalent to a strict linear inequality: $\sum_{i=1}^k p_i < 0.25$, where $k = \max\{i \mid a_i < \alpha\}$.
- (i) $\text{quartile}(x) \le \alpha$ is convex because it is equivalent to a linear inequality: $\sum_{i=k+1}^n p_i \ge 0.25$, where k is such that $a_k < \alpha \le a_{k+1}$ (define $k = 0$ if $\alpha \le a_1$).

9.5 Which of these functions are convex?

- (a) $f(x) = e^x - 1$ on $\mathbb{R}$ is **strictly convex** because its second derivative is positive everywhere. Therefore, it is also **quasiconvex**. It is also **quasiconcave** but not concave.
- (b) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$ is **neither convex nor concave** because its Hessian is neither positive nor negative semidefinite. It is **quasiconcave** since its superlevel sets $\{(x_1, x_2) \mid x_1 x_2 \ge \alpha\}$ are convex. It is not quasiconvex.
- (c) $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbb{R}^2_{++}$ is **convex** because its Hessian is positive semidefinite. Therefore, it is also **quasiconvex**. It is not concave or quasiconcave.
- (d) $f(x_1, x_2) = x_1/x_2$ on $\mathbb{R}^2_{++}$ is **neither convex nor concave** because its Hessian is indefinite. It is **quasiconvex and quasiconcave** (quasilinear), since both sublevel and superlevel sets are halfspaces.
- (e) $f(x_1, x_2) = x_1^2/x_2$ on $\mathbb{R} \times \mathbb{R}_{++}$ is **convex** because its Hessian is positive semidefinite. Therefore, it is also **quasiconvex**. It is not concave or quasiconcave.
- (f) $f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$, $0 \le \alpha \le 1$, on $\mathbb{R}^2_{++}$ is **concave and quasiconcave** because its Hessian is negative semidefinite. It is not convex or quasiconvex.
- (g) $f(x) = \left(\sum_{i=1}^n x_i^p\right)^{1/p}$ with $\text{dom} f = \mathbb{R}^n_{++}$ and $p < 1$, $p \ne 0$ is **concave**. This includes, for example, the harmonic mean and the p-norm with $p < 1$.
- 3.18 (a) $f(X) = \text{tr}X^{-1}$ is **convex** on $\text{dom} f = S^n_{++}$ because for any $Z \succ 0$ and $V \in S^n$, $g(t) = \text{tr}(Z + tV)^{-1}$ is a positive weighted sum of convex functions $1/(1 + t\lambda_i)$.
- 3.18 (b) $f(X) = (\det X)^{1/n}$ is **concave** on $\text{dom} f = S^n_{++}$ because $g(t) = (\det(Z + tV))^{1/n}$ is a geometric mean of affine functions, which is concave.
- 3.19 (a) $f(x) = \sum_{i=1}^r \alpha_i x_{[i]}$ is **convex** if $\alpha_1 \ge \cdots \ge \alpha_r \ge 0$, since it is a nonnegative sum of convex functions $x_{[1]}, x_{[1]} + x_{[2]}, \ldots, x_{[1]} + \cdots + x_{[r]}$.
- 3.19 (b) $f(x) = -\int_0^{2\pi} \log T(x, \omega) d\omega$, where $T(x, \omega) = x_1 + x_2 \cos\omega + \cdots + x_n \cos((n-1)\omega)$, is **convex** on $\{x \mid T(x, \omega) > 0,\ 0 \le \omega \le 2\pi\}$ because $g(x, \omega) = -\log(T(x, \omega))$ is convex in x for fixed ω, and the integral preserves convexity.
- 3.20 (a) $f(x) = \|Ax - b\|$ is **convex** because it is the composition of a convex norm and an affine function.
- 3.20 (b) $f(x) = -(\det(A_0 + x_1 A_1 + \cdots + x_n A_n))^{1/m}$ is **convex** on $\{x \mid A_0 + x_1 A_1 + \cdots + x_n A_n \succ 0\}$ because $h(X) = -(\det X)^{1/m}$ is convex and the argument is affine in x.
- 3.20 (c) $f(x) = \text{tr}(A_0 + x_1 A_1 + \cdots + x_n A_n)^{-1}$ is **convex** on $\{x \mid A_0 + x_1 A_1 + \cdots + x_n A_n \succ 0\}$ because $\text{tr}X^{-1}$ is convex and the argument is affine.
- 3.21 (a) $f(x) = \max_{i=1,\ldots,k} \|A^{(i)}x - b^{(i)}\|$ is **convex** because it is the pointwise maximum of convex functions.
- 3.21 (b) $f(x) = \sum_{i=1}^r |x|_{[i]}$ is **convex** because it can be written as the pointwise maximum of convex functions (sums of the r largest absolute values).
- 3.22 (a) $f(x) = -\log(-\log(\sum_{i=1}^m e^{a_i^T x + b_i}))$ is **convex** on $\{x \mid \sum_{i=1}^m e^{a_i^T x + b_i} < 1\}$ because it is the composition of a convex function and affine mappings, and preserves convexity.
- 3.22 (b) $f(x, u, v) = -\sqrt{uv - x^T x}$ on $\{(x, u, v) \mid uv > x^T x, u, v > 0\}$ is **convex** because $-\sqrt{x_1 x_2}$ is convex on $\mathbb{R}^2_{++}$ and decreasing in each argument, and $uv - x^T x$ is concave.
- 3.22 (c) $f(x, u, v) = -\log(uv - x^T x)$ on $\{(x, u, v) \mid uv > x^T x, u, v > 0\}$ is **convex** because $-\log$ is convex and decreasing, and $uv - x^T x$ is concave.
- 3.22 (d) $f(x, t) = -(t^p - \|x\|_p^p)^{1/p}$ for $p > 1$ and $\{(x, t) \mid t \ge \|x\|_p\}$ is **convex** because it is the composition of a convex and decreasing function and concave functions.
- 3.22 (e) $f(x, t) = -\log(t^p - \|x\|_p^p)$ for $p > 1$ and $\{(x, t) \mid t > \|x\|_p\}$ is **convex** because it is the composition of a convex and decreasing function and a concave function.